Spin(7)-INSTANTONS, STABLE BUNDLES AND THE BOGOMOLOV INEQUALITY FOR COMPLEX 4-TORI

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Abstract. Using gauge theory for Spin(7) manifolds of dimension 8, we develop a procedure, called Spin-rotation, which transforms a (stable) holomorphic structure on a vector bundle over a complex torus of dimension 4 into a new holomorphic structure over a different complex torus. We show non-trivial examples of this procedure by rotating a decomposable Weil abelian variety into a non-decomposable one. As a byproduct, we obtain a Bogomolov type inequality, which gives restrictions for the existence of stable bundles on an abelian variety of dimension 4, and show examples in which this is stronger than the usual Bogomolov inequality.

1. Introduction

Let M be a smooth four-dimensional compact oriented manifold, and let $E \to M$ be a hermitian complex vector bundle. If M is endowed with a Kähler structure, then the Hitchin-Kobayashi correspondence gives a one-to-one correspondence between (poly)stable holomorphic structures on E and anti-self-dual connections (instantons) on E. A Kähler structure on M is basically a Riemannian metric with holonomy in $\mathrm{U}(2) \subset \mathrm{SO}(4)$. Under this inclusion, the Hermite-Einstein equation is translated into the anti-self-dual equation. Therefore, if we know of the existence of anti-self-dual connections for E (e.g. if the Donaldson invariant corresponding to E is non-zero, which guarantees the non-emptiness of the moduli space of anti-self-dual connections), then for *any* Kähler structure ω on M, E admits an structure of polystable holomorphic bundle. In particular, if (M, ω) is a projective surface, then $c_1(E)$ is an algebraic class. However, this does not say anything new about the algebraicity of Hodge classes for complex projective surfaces, since the Hodge Conjecture is known for $(1,1)$-classes (cf. Lefschetz theorem on $(1,1)$-classes [5]).

In [17], M. Toma uses the above idea to find stable bundles with given Chern classes on a Kähler 4-torus Y. By what we said above, the condition to have a (poly)stable holomorphic structure is a condition on the Riemmanian structure of Y. So one may vary the inclusion $\mathrm{U}(2) \subset \mathrm{SO}(4)$, hence changing the Kähler structure on Y without

2000 *Mathematics Subject Classification.* Primary: 14D21. Secondary: 53C07, 14K22, 32G20.

Key words and phrases. Spin(7)-instanton, stable bundle, Bogomolov inequality, abelian variety, period matrices.

Partially supported through grant MICINN (Spain) MTM2010-17389.

changing the Riemannian metric. Under these circumstances, the existence of a stable holomorphic structure on a bundle E for one Kähler structure ω on Y implies the same for another Kähler structure ω'. A similar approach is used in [8] for studying stable bundles on hyperkähler and hypercomplex manifolds.

In eight dimensions, we can try to follow the same scheme, by considering the inclusion $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$. The proposal to develop gauge theoretic methods in 6, 7 and 8 dimensions appears in the seminal paper of S. Donaldson and R. Thomas [3]. In dimension 8, it is natural to consider a (compact, oriented) manifold M with a Spin(7)-structure (that is, endowed with a Riemannian metric whose holonomy lies in $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$). For such M, the notion of Spin(7)-instanton is defined in [3]. Ideally, this should give rise to moduli spaces of Spin(7)-instantons and associated invariants (in the vein of the Donaldson invariants in four dimensions). Basic properties of the linearized Spin(7)-instanton equation are considered in [13]. Construction of non-trivial examples of Spin(7)-instantons are given in the D. Phil. thesis of C. Lewis [6]. The deep analysis of the bubbling properties of Spin(7)-instantons is undertaken by G. Tian in [16]. However, the transversality issues related to the construction of the moduli spaces of Spin(7)-instantons seem to be out of reach at present.

An 8-dimensional compact Riemannian manifold M with holonomy in SU(4) is a manifold with a Calabi-Yau structure (a Kähler manifold (M,ω) endowed with a global holomorphic everywhere non-zero $(4,0)$-form θ). Let $E \to M$ be an $\mathrm{SU}(r)$-vector bundle. A Spin(7)-instanton on E is equivalent to a polystable holomorphic structure under the obvious topological condition $c_2(E) \in H^{2,2}(M)$. This result already appears in [6]. G. Tian points out [16] that this could be used to prove the Hodge conjecture if there is a way to prove the existence of Spin(7)-instantons on E. The observation appears very explicitly in [12], where an (unsuccessful) attempt to construct Spin(7)-instantons for some abelian varieties of Weil type, for which the Hodge Conjecture is yet unknown, is proposed.

In this paper we study Spin(7)-instantons for Kähler complex tori of dimension 4. We consider SU(4)-structures giving rise to the same (flat) Spin(7)-structure on an 8-torus $X = \mathbb{R}^8/\Lambda$, determining under which circumstances the existence of a (poly)stable holomorphic structure on $E \to X$ for (X,ω,θ) produces a (poly)stable holomorphic structure for (X,ω',θ') (cf. Theorem 13). We call the process of going from (ω,θ) to (ω',θ') a Spin-*rotation*, as it is given by conjugating the inclusion $\mathrm{SU}(4) \hookrightarrow \mathrm{Spin}(7)$ by an element in Spin(7). The reason that this is realisable hinges on the fact that we do not try to prove the non-emptiness of the moduli spaces of Spin(7)-instantons by relating them for different metrics, we just use that the Spin(7)-structure of X is fixed, so a given Spin(7)-instanton keeps satisfying the Spin(7)-instanton equation while performing a Spin-rotation.

The Hodge Conjecture predicts that Hodge classes (rational (p,p)-classes) on a projective complex n-manifold are represented by (rational) algebraic cycles (see [7] for a nice account on the subject). The Hodge Conjecture is known to be true for dimension $n \leq 3$ and for $p = 1, n-1$. The first instance in which it is still open is $n = 4$, $p = 2$. A. Weil proposed [19] a family of abelian varieties of dimension 4 (based on an example of Mumford [11]) to be studied as possible counterexamples to the Hodge Conjecture. These abelian varieties have complex multiplication by $L = \mathbb{Z}[\sqrt{-d}]$, $d > 0$ a square-free integer, and a subspace of Hodge $(2,2)$-classes (now known as *Weil classes*) which are not products of $(1,1)$-classes, and therefore potentially non-algebraic. Surprisingly, for the case of complex multiplication by $\mathbb{Z}[\sqrt{-d}]$ with $d = 1, 3$, C. Schoen [15] proved that the generic Weil abelian variety satisfies the Hodge Conjecture, by constructing the algebraic cycles representing the Weil classes (the method of proof is algebro-geometric and not extendable to other values of d). See [10] for an account on the state of the Hodge Conjecture for abelian four-folds.

One possible route to construct algebraic cycles for a Weil abelian variety is to construct stable bundles with given Chern classes. The Spin-rotations are a new method to construct stable bundles on complex tori, and has the nice feature of being a highly non-algebro-geometric proceduce. We show an example where Spin-rotation is performed starting with an abelian variety of Weil type which is a product of two abelian surfaces. The resulting Spin-rotated torus is another Weil abelian variety which turns out to be non-decomposable (see Section 8), very different from the initial one from the algebro-geometric and arithmetic points of view. This is a deep indication that the Spin-rotation have a very non-trivial effect.

On the negative side, C. Voisin [18] gave examples of Kähler 4-tori which are not algebraic for which the Hodge Conjecture does not hold. These tori are of Weil type, and the non-existence of complex cycles is proved via the non-existence of (stable) holomorphic structures on vector bundles, which in turn hinges on the Bogomolov inequality. The Spin-rotations give an explanation to the phenomenon in [18]. For a bundle to be stable, it has to be positive enough, so the examples of [18] cannot be Spin-rotated to abelian varieties.

Actually, we get non-trivial restrictions for the existence of stable bundles with given Chern classes, producing a Bogomolov type inequality:

Theorem. *Let (X,ω) be an abelian four-fold. Let $\beta_0 \in H^{2,2}_{prim}(X)$ be a rational primitive $(2,2)$-class, and define*

$$k_m(\beta_0) = \frac{1}{4} \max\{\beta_0 \wedge c \wedge \bar{c} \,|\, c \in H^{2,0}(X), |c|_\omega = 1\}.$$

Then, for any ω-polystable bundle E, such that the class $\beta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2$ has component in $H^{2,2}_{prim}(X)$ equal to β_0, we have

$$\beta(E) \cup [\omega]^2 \geq k_m(\beta_0)\, [\omega]^4 \,.$$

Another consequence is the following inequality

$$\frac{||c||_Z^2}{\mathrm{vol}(Z)} \leq \frac{1}{2} \frac{||c||_X^2}{\mathrm{vol}(X)} \,,$$

for any $Z \subset X$ surface which is a local complete intersection and $c \in A_+ \subset H^{2,0}(X)$ (see Section 2 below for the definition of A_+).

Acknowledgements: I am indebted to many people for very useful conversations, suggestions and encouragement. In chronological order, I have to mention Gang Tian, Ludmil Katzarkov, Simon Donaldson, Richard Thomas, Claire Voisin, Ignasi Mundet, Francisco Presas, Javier Fernández de Bobadilla, Javier Martínez, Luis Solá, Matei Toma, Philippe Eyssidieux and Adrian Langer. Also I want to thank Dominic Joyce for providing me with a copy of [6].

2. Spin(7) geometry

2.1. **The group** Spin(7)**.** Consider $\mathbb{R}^8$ with coordinates $(x_1, \ldots, x_8)$, and the following 4-form:

$$\begin{aligned}\Omega_0 = {} & dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} - dx_{1458} - dx_{1467} \\ & - dx_{2358} - dx_{2367} - dx_{2457} + dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678} \,,\end{aligned} \tag{1}$$

where we abbreviate $dx_{ij\ldots k} = dx_i \wedge dx_j \wedge \ldots \wedge dx_k$. The group Spin(7) is the subgroup of $\mathrm{GL}_+(8, \mathbb{R})$ which leaves invariant Ω_0. An Spin(7)-structure on an oriented vector space of dimension 8 is the choice of a 4-form Ω which can be written as (1) in a suitable oriented frame.

Given Ω, there is a well-defined scalar product g. This holds because of the inclusion $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. The Hodge star operator $*$ associated to g makes Ω self-dual, $*\,\Omega = \Omega$. The standard volume form is $\mathrm{vol} = dx_{12\ldots 8}$. Then $\Omega \wedge \Omega = 14\,\mathrm{vol}$.

We shall use some representation theory of Spin(7). The standard representation is $V = \mathbb{R}^8$ with the action given by $\mathrm{Spin}(7) \hookrightarrow \mathrm{GL}(8, \mathbb{R})$. There is a 7-dimensional spin representation $S = \mathbb{R}^7$ given by the double cover $\mathrm{Spin}(7) \to \mathrm{SO}(7) \subset \mathrm{GL}(7, \mathbb{R})$. Let $\bigwedge^i = \bigwedge^i V$, $1 \leq i \leq 8$. Then we have the following decomposition of the

representations $\bigwedge^i$ into irreducible factors:

$$
\begin{aligned}
&\bigwedge^1 = \bigwedge^1_8 \\
&\bigwedge^2 = \bigwedge^2_7 \oplus \bigwedge^2_{21} \\
&\bigwedge^3 = \bigwedge^3_8 \oplus \bigwedge^3_{48} \\
&\bigwedge^4 = \bigwedge^4_+ \oplus \bigwedge^4_-, \qquad \text{where } \bigwedge^4_+ = \bigwedge^4_1 \oplus \bigwedge^4_7 \oplus \bigwedge^4_{27}, \quad \bigwedge^4_- = \bigwedge^4_{35},
\end{aligned}
$$

where we write $\bigwedge^i_j$ for an irreducible sub-representation of $\bigwedge^i$ of dimension j.

These are characterized as follows. $\bigwedge^2_7$ is isomorphic to the spin representation S and consists of those $\alpha \in \bigwedge^2$ such that $*(\Omega \wedge \alpha) = 3\alpha$ (equivalently, $\alpha \in \bigwedge^2_7$ if and only if $\Omega \wedge \alpha \wedge \alpha = 3|\alpha|^2$). The summand $\bigwedge^2_{21}$ is characterised by the equation $*(\Omega \wedge \alpha) = -\alpha$.

The summand $\bigwedge^3_8$ consists of elements of the form $*(\Omega \wedge \phi)$, $\phi \in \bigwedge^1$. And $\bigwedge^3_{48}$ is its orthogonal complement, i.e. the kernel of $\wedge\Omega : \bigwedge^3 \to \bigwedge^7$.

The Hodge operator $*$ acts as ± 1 on $\bigwedge^4_\pm$. The summand $\bigwedge^4_1$ is generated by Ω. The summand $\bigwedge^4_7$ is isomorphic to S, and it is the image under the chain of maps $S \subset \bigwedge^2 V \subset V \otimes V \cong \bigwedge^1_8 \otimes \bigwedge^3_8 \xrightarrow{\wedge} \bigwedge^4$. The piece $\bigwedge^4_{27} \cong \mathrm{Sym}^2_0 S$, the traceless part of the symmetric product, and it appears as the image under $\mathrm{Sym}^2_0 S \subset S \otimes S \cong \bigwedge^2_7 \otimes \bigwedge^2_7 \xrightarrow{\wedge} \bigwedge^4$. Finally, $\bigwedge^4_{35} \cong \bigwedge^3 S$ under $\bigwedge^3 S \subset S \otimes \bigwedge^2 S \cong \bigwedge^2_7 \otimes \bigwedge^2_{21} \xrightarrow{\wedge} \bigwedge^4$.

By the discussion above, the wedge product gives an isomorphism $\mathrm{Sym}^2 \bigwedge^2_7 \cong \bigwedge^4_{27} \oplus \bigwedge^4_1$. Also the wedge product goes as $\bigwedge^2_7 \bigotimes \bigwedge^2_{21} \longrightarrow \bigwedge^4_7 \oplus \bigwedge^4_{35}$, surjectively.

Let M be a compact smooth oriented manifold of dimension 8. A Spin(7)-structure on M is the choice of a 4-form $\Omega \in \Omega^4(M)$ such that:

- Ω_p is a Spin(7)-structure on T_pM for each $p \in M$. The Spin(7)-structure induces a Riemannian metric g.
- $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection associated to g. This is equivalent to Ω being closed and co-closed [14], $d\Omega = d * \Omega = 0$. But as Ω is self-dual, it is equivalent to $d\Omega = 0$.

Note that $[\Omega] \in H^4(M)$ is non-trivial.

If Ω is a Spin(7)-structure, we say that (M, Ω) is a Spin(7)-manifold. Note that this is also equivalent to having a Riemannian metric g on M such that its (restricted) holonomy group satisfies $\mathrm{Hol}_g \subset \mathrm{Spin}(7)$: choose Ω_{p_0} invariant by Spin(7) at some $p_0 \in M$, and parallel transport it by the connection to get a global Ω. However, note that we shall understand that a Spin(7)-manifold is a Riemannian manifold with holonomy contained in Spin(7) *together with* a chosen 4-form as above.

For a Spin(7)-manifold, the Laplacian on forms leaves invariant the bundles $\bigwedge^i_j(TM)$. So it induces a decomposition on harmonic forms as $\mathcal{H}^i(M) = \bigoplus \mathcal{H}^i_j(M)$, accordingly. However, note that $\mathcal{H}^i_j(M)$ are not of dimension j, and they are not Spin(7)-representations (cf. [4]).

2.2. **Relationship between** SU(4) **and** Spin(7)**.** Let V be a complex vector space of dimension 4. This can be understood as a real vector space of dimension 8, $V = \mathbb{R}^8$ together with an endomorphism $J : V \to V$, such that $J^2 = -\operatorname{Id}$. This corresponds to the inclusion $\mathrm{GL}(4,\mathbb{C}) \subset \mathrm{GL}_+(8,\mathbb{R})$. An SU(4)-structure is the same as the choice of:

- an hermitian metric h on V. Correspondingly, $g = \mathrm{Re}(h)$ is a scalar product compatible with J (i.e. $g(Jx, Jy) = g(x, y)$), and $\omega(x, y) = g(x, Jy)$ is a $(1,1)$-form, also compatible with J. This corresponds to $\mathrm{U}(4) \subset \mathrm{SO}(8)$.
- a non-zero $(4,0)$-form θ. This produces a trivialization of $\bigwedge^{4,0} V$.

There is a complex frame $(z_1, \ldots, z_4) = (x_1 + ix_2, \ldots, x_7 + ix_8)$ such that

$$\begin{aligned}\omega &= \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \ldots + dz_4 \wedge d\bar{z}_4) = dx_{12} + dx_{34} + dx_{56} + dx_{78}\,,\\ \theta &= dz_1 \wedge \ldots \wedge dz_4\,.\end{aligned}$$

The group U(4) is the subgroup of $\mathrm{GL}_+(8,\mathbb{R})$ which leaves ω, J fixed (alternatively, ω, g fixed). The group SU(4) leaves ω, J and θ fixed. Note that $\mathrm{U}(4) \subset \mathrm{SO}(8)$.

There is an inclusion $\mathrm{SU}(4) \subset \mathrm{Spin}(7)$, so an SU(4)-structure induces a Spin(7)-structure. This holds by choosing the following real 4-form:

$$\Omega = \frac{1}{2}\omega \wedge \omega + \mathrm{Re}(\theta). \tag{2}$$

It is easy to see that if we write $z_1 = x_1 + ix_2, \ldots, z_4 = x_7 + ix_8$, then (2) equals (1).

Note that $\theta \wedge \bar{\theta} = 16\,\mathrm{vol}$, $\mathrm{Re}(\theta) \wedge \mathrm{Re}(\theta) = \frac{1}{2}\theta \wedge \bar{\theta} = 8\,\mathrm{vol}$ and $\omega^4 = 24\,\mathrm{vol}$. In particular, as $*\theta = \bar{\theta}$, we have that $|\theta| = 4$, $|\mathrm{Re}(\theta)| = 2\sqrt{2}$ and $|\omega| = 2$.

For $\alpha \in \bigwedge^2_7$, we have $\Omega \wedge \alpha \wedge \alpha = \langle *(\Omega \wedge \alpha), \alpha\rangle = \langle 3\alpha, \alpha\rangle = 3|\alpha|^2$. As

$$\Omega \wedge \omega \wedge \omega = \frac{1}{2}\omega^4 = 12\,\mathrm{vol} = 3|\omega|^2,$$

we have that $\omega \in \bigwedge^2_7$. Fixing a Spin(7)-structure, the compatible SU(4)-structures (those inducing the given Spin(7)-structure) are parametrized by the homogeneous space

$$P = \mathrm{Spin}(7)/\,\mathrm{SU}(4)\,.$$

Here the group Spin(7) acts on the subgroups $G < \mathrm{Spin}(7)$ which are isomorphic to SU(4), by conjugation.

Lemma 1. *P is diffeomorphic to $S(\bigwedge^2_7)$, which is a 6-sphere.*

Proof. The elements of P can be understood as pairs (ω, θ) satisfying (2), where Ω is fixed. Let $P' = S(\bigwedge^2_7)$, the sphere of radius 2. We consider the map $\Phi : P \to P' = S(\bigwedge^2_7)$, $\Phi(\omega, \theta) = \omega$. This is well-defined, since $\omega \in \bigwedge^2_7$ and $|\omega| = 2$. The map is clearly Spin(7)-equivariant, and if $\phi \in \mathrm{Spin}(7)$ leaves ω fixed, then it fixes $\mathrm{Re}(\theta)$ as well. Therefore it fixes θ which is the $(4,0)$-component of $\mathrm{Re}(\theta)$ (J being fixed as well). So Φ is injective.

The dimension of P is $\dim P = \dim \mathrm{Spin}(7) - \dim \mathrm{SU}(4) = 21 - 15 = 6$, so P and P' are both smooth 6-dimensional manifolds. Thus Φ is regular, hence a local diffeomorphism, and so a covering. This implies that it is a diffeomorphism by simply-connectivity of the sphere. $\square$

Lemma 1 says that given $\omega \in S(\bigwedge^2_7)$, with $|\omega| = 2$, then we have a well-defined SU(4)-structure, where $\mathrm{Re}(\theta) = \Omega - \frac{1}{2}\omega^2$, the complex structure J is defined by $g(x, y) = \omega(Jx, y)$, and $\mathrm{Im}(\theta)(u_1, u_2, u_3, u_4) = \mathrm{Re}(\theta)(Ju_1, u_2, u_3, u_4)$.

The irreducible real representations of SU(4) are as follows. Let V be the standard representation of SU(4), that is $V = \mathbb{R}^8$ with the action given by $\mathrm{SU}(4) \subset \mathrm{GL}(4, \mathbb{C}) \hookrightarrow \mathrm{GL}(8, \mathbb{R})$. Let $\bigwedge^k_{\mathbb{C}} = (\bigwedge^k V) \otimes \mathbb{C}$, $1 \leq k \leq 8$. The action of J gives a decomposition

$$\bigwedge\nolimits^k_{\mathbb{C}} = \bigoplus_{\substack{i+j=k \\ 0 \leq i,j \leq 4}} \bigwedge\nolimits^{i,j} V \, .$$

We have the isomorphism $\bigwedge^{i,j} V \cong \overline{\bigwedge^{j,i} V}$. For $i < j$, we denote

$$\triangle^{i,j} = \mathrm{Re}(\bigwedge\nolimits^{i,j} \oplus \bigwedge\nolimits^{j,i}),$$

which is a real representation, whose complexification is $\bigwedge^{i,j} \oplus \bigwedge^{j,i}$. It is irreducible and of (real) dimension $2\binom{4}{i}\binom{4}{j}$. Its elements are of the form $\frac{1}{2}(\alpha + \bar{\alpha})$, $\alpha \in \bigwedge^{i,j}$. For given i, we denote

$$\triangle^{i,i} = \mathrm{Re}(\bigwedge\nolimits^{i,i}),$$

which is irreducible of (real) dimension $\binom{4}{i}^2$.

The choice of $\omega \in \bigwedge^{1,1}$ gives a further decomposition. For $i + j \leq 4$,

$$\bigwedge\nolimits^{i,j} = \bigoplus_{r=m}^{M} \bigwedge\nolimits^{i-r,j-r}_{prim} \cdot \omega^r \, ,$$

where $m = \max\{0, i + j - 4\}$, $M = \min\{i, j\}$, and the primitive components are defined by

$$\bigwedge\nolimits^{a,b}_{prim} = \ker(\omega^{5-(a+b)} : \bigwedge\nolimits^{a,b} \to \bigwedge\nolimits^{5-b,5-a}),$$

for $a+b \leq 4$. As ω is a real form, we have also a decomposition of the real representations

$$\triangle^{i,j} = \bigoplus_{r=m}^{M} \triangle_{prim}^{i-r,j-r} \cdot \omega^r \,,$$

where $\triangle_{prim}^{a,b}$ is defined in the obvious manner.

The element θ gives an extra decomposition. There is a complex linear map [3]

$$L : \bigwedge^{2,0} \longrightarrow \bigwedge^{0,2} ,$$

determined uniquely by

$$\alpha \wedge \overline{L(\alpha)} = \frac{1}{4} |\alpha|^2 \, \theta.$$

As $|\theta| = 4$, the map L is isometric. Extend L to $L : \bigwedge^{0,2} \to \bigwedge^{2,0}$, via $L(\bar{\alpha}) = \overline{L(\alpha)}$. So $\alpha \wedge \overline{L(\alpha)} = \frac{1}{4}|\alpha|^2 \bar{\theta}$, for $\alpha \in \bigwedge^{0,2}$. Thus L gives an endomorphism of $A = \triangle^{2,0} = \mathrm{Re}(\bigwedge^{2,0} \oplus \bigwedge^{0,2})$. It is easy to see that $L^2 = \mathrm{Id}$, so that L gives a decomposition of A into (± 1)-eigenspaces. Write $A = A_+ \oplus A_-$. Both A_+, A_- are real representations of dimension 6.

Finally, $\triangle^{4,0}$ is of dimension 2, and it decomposes as $\langle \mathrm{Re}(\theta) \rangle \oplus \langle \mathrm{Im}(\theta) \rangle$. To sum up, we have the following decomposition of the representations $\bigwedge^i$ into irreducible summands:

$$\begin{aligned}
\bigwedge^1 &= \triangle^{1,0}, \\
\bigwedge^2 &= A_+ \oplus A_- \oplus \triangle_{prim}^{1,1} \oplus \langle \omega \rangle, \\
\bigwedge^3 &= \triangle^{3,0} \oplus \triangle_{prim}^{2,1} \oplus \triangle^{1,0} \, \omega, \\
\bigwedge^4 &= \langle \mathrm{Re}(\theta) \rangle \oplus \langle \mathrm{Im}(\theta) \rangle \oplus \triangle_{prim}^{3,1} \oplus \triangle_{prim}^{2,2} \oplus A_+ \omega \oplus A_- \omega \oplus \triangle_{prim}^{1,1} \omega \oplus \langle \omega^2 \rangle.
\end{aligned}$$

The relationship of the irreducible representations of $\mathrm{SU}(4)$ and $\mathrm{Spin}(7)$ is given by the following result.

Proposition 2. *Let V be an 8-dimensional vector space with an $\mathrm{SU}(4)$-structure, and consider the induced $\mathrm{Spin}(7)$-structure. Then the irreducible $\mathrm{Spin}(7)$-representations*

decompose into SU(4)*-representations as follows:*

$$\begin{aligned}
&\textstyle\bigwedge^2_7 = \langle\omega\rangle \oplus A_+ \,, \\
&\textstyle\bigwedge^2_{21} = \triangle^{1,1}_{prim} \oplus A_- \,, \\
&\textstyle\bigwedge^3_8 = \triangle^{3,0} \oplus \triangle^{1,0}\omega \,, \\
&\textstyle\bigwedge^3_{48} = \triangle^{2,1}_{prim} \,, \\
&\textstyle\bigwedge^4_1 = \langle\Omega\rangle = \langle \frac{1}{2}\omega^2 + \mathrm{Re}(\theta)\rangle \,, \\
&\textstyle\bigwedge^4_7 = A_-\omega \oplus \langle \mathrm{Im}(\theta)\rangle \,, \\
&\textstyle\bigwedge^4_{27} = A_+\omega \oplus \triangle^{2,2}_{prim} \oplus \langle \omega^2 - \frac{3}{2}\mathrm{Re}(\theta)\rangle \,, \\
&\textstyle\bigwedge^4_{35} = \triangle^{1,3}_{prim} \oplus \triangle^{1,1}_{prim}\omega \,.
\end{aligned}$$

Proof. We already know that $\omega \in \bigwedge^2_7$. Let $a \in A_\pm$. Then $a = \frac{1}{2}(\alpha+\bar\alpha)$ and $L(a) = \pm a$. So $L(\alpha) = \pm\bar\alpha$. Therefore

$$\begin{cases} \alpha\wedge\bar\alpha\wedge\omega^2 = 2|\alpha|^2\mathrm{vol} \\ \alpha\wedge\alpha\wedge\frac{1}{4}\bar\theta = \pm\alpha\wedge\overline{L(\alpha)}\wedge\frac{1}{4}\bar\theta = \pm|\alpha|^2\,\frac{1}{4}\theta\wedge\frac{1}{4}\bar\theta = \pm|\alpha|^2\mathrm{vol} \end{cases}$$

the first equality being always true for $(2,0)$-forms. So, using that $a = \frac{1}{2}(\alpha+\bar\alpha)$,

$$\begin{cases} a\wedge a\wedge\frac{1}{2}\omega^2 = \frac{1}{2}\alpha\wedge\bar\alpha\wedge\frac{1}{2}\omega^2 = \frac{1}{2}|\alpha|^2\mathrm{vol} = |a|^2\mathrm{vol} \\ a\wedge a\wedge\mathrm{Re}(\theta) = \frac{1}{8}(\alpha\wedge\alpha\wedge\bar\theta + \bar\alpha\wedge\bar\alpha\wedge\theta) = \pm|\alpha|^2\mathrm{vol} = \pm 2|a|^2\mathrm{vol} \end{cases}$$

We compute

$$\begin{aligned} a\wedge a\wedge\Omega &= a\wedge a\wedge\left(\frac{1}{2}\omega^2 + \mathrm{Re}(\theta)\right) \\ &= (|a|^2 \pm 2|a|^2)\mathrm{vol}\,. \end{aligned}$$

So, for $a \in A_+$, $a\wedge a\wedge\Omega = 3|a|^2\mathrm{vol}$. Hence $A_+ \subset \bigwedge^2_7$.

For $a \in A_-$, we have $a\wedge a\wedge\Omega = -|a|^2\mathrm{vol}$, proving that $A_- \subset \bigwedge^2_{21}$. The decomposition of $\bigwedge^2_{21}$ now follows by dimensionality reasons.

The decompositions in the third and fourth lines also follow by dimensionality reasons.

The fifth line is by definition. Now the dimensions of $A_-\omega$, $A_+\omega$, $\triangle^{2,2}_{prim}$, $\triangle^{1,3}_{prim}$ and $\triangle^{1,1}_{prim}\omega$ are $6, 6, 20, 20$ and 15, respectively. Therefore the dimensions of the lines sixth to eighth should be $1+6, 1+20+6, 15+20$. This implies that $\triangle^{1,1}_{prim}\omega \subset \bigwedge^4_{35}$. Now recall that $\bigwedge^4_{35} = \bigwedge^4_-$. So to see that $\triangle^{1,3}_{prim} \subset \bigwedge^4_{35}$, it is enough to see that for

$\alpha \in \triangle^{1,3}_{prim}$ we have $\alpha\wedge\alpha = -|\alpha|^2\text{vol}$. Taking as example $\alpha = \mathrm{Re}(dz_1\wedge d\bar z_2\wedge d\bar z_3\wedge d\bar z_4)$, we check this. This proves the eighth line.

For the seventh line, recall that image of $\mathrm{Sym}^2(\bigwedge^2_7)$ (under the wedge product) is $\bigwedge^4_{27}\oplus\bigwedge^4_1$. This image contains $A_+\omega$ and ω^2. As it also contains Ω, we have $\mathrm{Re}(\theta)\in\bigwedge^4_{27}\oplus\bigwedge^4_1$. Since $\omega^2-\frac{3}{2}\mathrm{Re}(\theta)\perp\frac{1}{2}\omega^2+\mathrm{Re}(\theta)$, it must be $\omega^2-\frac{3}{2}\mathrm{Re}(\theta)\in\bigwedge^4_{27}$. Also $A_+\omega\subset\bigwedge^4_{27}$. The result follows.

For the sixth line, note that it must be $A_-\omega\subset\bigwedge^4_7$. The extra term must be $\mathrm{Im}(\theta)$, since $\mathrm{Im}(\theta)\perp\mathrm{Re}(\theta),\omega^2$. □

Remark 3. As we said in Subsection 2.1, there is an isomorphism $\mathrm{Sym}^2\bigwedge^2_7\cong\bigwedge^4_{27}\oplus\bigwedge^4_1$. Under it, we see that

$$\mathrm{Sym}^2 A_+\cong\Delta^{2,2}_{prim}\oplus\langle\mathrm{Re}(\theta)+8\omega^2\rangle. \tag{3}$$

Also, the surjection $\bigwedge^2_7\otimes\bigwedge^2_{21}\longrightarrow\bigwedge^4_7\oplus\bigwedge^4_{35}$ restrics to give a surjection

$$A_+\otimes A_-\twoheadrightarrow\Delta^{1,1}_{prim}\omega\oplus\langle\mathrm{Im}(\theta)\rangle. \tag{4}$$

More explicitly, take the following elements of $\bigwedge^{2,0}$

$$\begin{cases} c_1 = dz_{12}+dz_{34} \\ c_2 = i\,dz_{12}-i\,dz_{34} \\ c_3 = dz_{13}-dz_{24} \\ c_4 = i\,dz_{13}+i\,dz_{24} \\ c_5 = dz_{14}+dz_{23} \\ c_6 = i\,dz_{14}-i\,dz_{23} \end{cases} \qquad \begin{cases} c'_1 = i\,dz_{12}+i\,dz_{34} \\ c'_2 = dz_{12}-dz_{34} \\ c'_3 = i\,dz_{13}-i\,dz_{24} \\ c'_4 = dz_{13}+dz_{24} \\ c'_5 = i\,dz_{14}+i\,dz_{23} \\ c'_6 = dz_{14}-dz_{23} \end{cases}$$

Then $\gamma_j=\frac{1}{2}(c_j+\bar c_j)$ are an orthogonal basis for A_+, and $\gamma'_j=\frac{1}{2}(c'_j+\bar c'_j)$ are an orthogonal basis for A_-. We have $c_j\wedge c_j=8\frac{\theta}{4}$, $c_j\wedge\overline{c}'_j=8i\,\frac{\theta}{4}$, and $c_i\wedge c_j=0$, $c_i\wedge c'_j=0$, for $i\neq j$. The generators of Sym^2A_+ are $\gamma_i\wedge\gamma_j$, and the generators of the image of $A_+\otimes A_-$ are $\gamma_i\wedge\gamma'_j$. From here (3) and (4) follow easily.

Remark 4. The group $\mathrm{SU}(4)$ acts on $A_\pm\cong\mathbb{R}^6$, compatibly with $\mathrm{SU}(4)\cong\mathrm{Spin}(6)\twoheadrightarrow\mathrm{SO}(6)$. In particular, the action is transitive on the elements of fixed norm.

Now let M be a compact Calabi-Yau 4-fold. That is, M is a complex manifold of (complex) dimension 4, endowed with a Kähler metric g given by a Kähler form $\omega\in\Omega^{1,1}(M)$, and a holomorphic volume form $\theta\in\Omega^{4,0}(M)$. Both ω and θ are parallel with respect to the Levi-Civita connection ∇. Therefore the holonomy $\mathrm{Hol}_g\subset\mathrm{SU}(4)$.

Such M gets an induced Spin(7)-structure, given by the choice $\Omega=\frac{1}{2}\omega^2+\mathrm{Re}(\theta)$.

Conversely, let M be a manifold with a Spin(7)-structure. To get an SU(4)-structure, we need to choose a 2-form ω which is a section of the sphere bundle $S(\bigwedge^2_7(TM))$. This gives a complex structure J defined point-wise by Lemma 1, making M an almost-complex manifold. Then there is a $(4,0)$-form θ defined by

$\mathrm{Re}(\theta) = \Omega - \frac{1}{2}\omega^2$. If ω is closed and J is integrable then M is Kähler. In this case, θ is also closed, and we have the holonomy contained in SU(4), i.e., M is a Calabi-Yau 4-fold.

3. Spin(7)-connections and holomorphic bundles

3.1. Spin(7)**-connections.** Let M be an 8-dimensional Spin(7)-manifold. Let $E \to M$ be a complex bundle of rank r, and let $c_i = c_i(E)$ denote its Chern classes. We put a hermitian metric on E, so that E is a $\mathrm{U}(r)$-bundle.

Let $\mathcal{A}_E$ denote the space of $\mathrm{U}(r)$-connections on E such that $\mathrm{Tr}\ F_A \in \Omega^2(M)$ is harmonic, where for $A \in \mathcal{A}_E$, we denote the curvature $F_A \in \Omega^2(\mathrm{End}\ E)$.

Let $\mathfrak{su}_E \subset \mathrm{End}\,(E)$ be the associated bundle of skew-hermitian traceless endomorphisms, and denote

$$F_A^o := F_A - \frac{1}{r}(\mathrm{Tr}\ F_A)\,\mathrm{Id}\, \in \Omega^2(\mathfrak{su}_E).$$

By Chern-Weil theory,

$$\int_M \mathrm{Tr}\,(F_A^o \wedge F_A^o) = 8\pi^2 \left(c_2 - \frac{r-1}{2r} c^2 \right) .$$

We set

$$\beta = \beta(E) := \frac{1}{8\pi^2}[\mathrm{Tr}\,(F_A^o \wedge F_A^o)] = c_2 - \frac{r-1}{2r}c^2 \in H^4(M).$$

The decomposition $\Omega^2 = \Omega^2_7 \oplus \Omega^2_{21}$ gives rise to projections π_7 and π_{21}.

Definition 5. We say that A is a Spin(7)-*instanton* if it satisfies the Spin(7)-instanton equation

$$\pi_7(F_A^o) = 0\,. \tag{5}$$

We have, for $A \in \mathcal{A}_E$,

$$\int_M \mathrm{Tr}\,(F_A^o \wedge F_A^o) \wedge \Omega = 8\pi^2 \beta \cup [\Omega]\,.$$

Also

$$\int_M \mathrm{Tr}\,(F_A^o \wedge F_A^o) \wedge \Omega = ||\pi_{21} F_A^o||^2 - 3||\pi_7 F_A^o||^2\,,$$

Here we have used that $\mathrm{Tr}\,(F_A^o \wedge F_A^o) = -\,\mathrm{Tr}\,((\overline{F_A^o})^t \wedge F_A^o)$, taking the transpose conjugate on matrices, and using $\alpha\wedge\alpha\wedge\Omega = 3|\alpha|^2\mathrm{vol}$ on $\bigwedge^2_7$, and $\alpha\wedge\alpha\wedge\Omega = -|\alpha|^2\mathrm{vol}$ on $\bigwedge^2_{21}$.

We have the Yang-Mills functional

$$\mathcal{F}(A) = ||F_A^o||^2 = ||\pi_{21}(F_A^o)||^2 + ||\pi_7(F_A^o)||^2\,.$$

The minimum of the functional $\mathcal{F}$ is attained for Spin(7)-instantons. Such minimum is the topological invariant $[8\pi^2(c_2 - \frac{r-1}{2r}c_1^2)] \cup [\Omega]$. The Yang-Mills functional is gauge-invariant, under the gauge group $\mathcal{G} = \mathrm{Aut}\,(E)$. In [13], Reyes-Carrión studies the linearization of the Spin(7)-equation, which is the elliptic complex

$$\Omega^0(\mathfrak{su}_E) \to \Omega^1(\mathfrak{su}_E) \to \Omega^2_7(\mathfrak{su}_E)\,.$$

There should be a moduli space of Spin(7)-instantons, but it has not been constructed yet. In [16] it is discussed how the Spin(7)-connections may blow-up for a sequence of Spin(7)-instantons, which is needed to compactify the moduli space. However, the regularity of the moduli spaces (transversality of the Spin(7)-equation) is a difficult issue, since the possible perturbations of metrics keeping the holonomy in Spin(7) are too scarce.

In this paper, we shall use the equation (5) as it is, without changing the Spin(7)-structure but changing the underlying SU(4)-structure. This does not affect the property of (5) having solutions. We will call this procedure Spin-*rotation*.

3.2. **Holomorphic structures.** Let (M, ω) be a Kähler manifold, that is, a complex manifold endowed with a U(4)-structure. Suppose that $E \to M$ is a complex rank r bundle with a hermitian metric, and let A be a U(r)-connection. We say that the connection A is Hermitian-Yang-Mills (w.r.t. ω) if

$$\begin{cases} F_A \in \Omega^{1,1}(\mathrm{End}\ E)\,, \\ \frac{i}{2\pi}\langle F_A, \omega\rangle = \lambda\ \mathrm{Id}\ , \end{cases}$$

where λ is a constant. This constant is determined by

$$\frac{i}{2\pi}\int \mathrm{Tr}\ F_A \wedge \omega^3 = [c_1(E)] \cup [\omega]^3 = \lambda[\omega]^4\,.$$

For a ω-HYM connection, we have $F_A^{0,2} = \bar\partial_A^2 = 0$, so $\bar\partial_A$ defines a holomorphic structure on E.

Proposition 6. *Let (M, ω) be a Kähler manifold. The following are equivalent:*

- *E admits a Hermitian-Yang-Mills connection w.r.t. ω.*
- *E admits a holomorphic structure $\bar\partial_A$ and $(E, \bar\partial_A)$ is polystable w.r.t. ω.*

$\square$

Here let (M, ω) be a Kähler manifold, denote $H = [\omega] \in H^2(M)$ the class of the Kähler form. We define

$$\deg_H(E) = \langle c_1(E) \cup [\omega]^3, [M]\rangle\,.$$

We define the slope of a bundle E as

$$\mu(E) := \frac{\deg_H(E)}{\mathrm{rk}\,(E)}\,.$$

A holomorphic bundle $\mathcal{E} = (E, \bar{\partial}_A)$ is stable if $\mu(\mathcal{F}) < \mu(\mathcal{E})$ for any proper holomorphic subsheaf $\mathcal{F} \subset \mathcal{E}$. It is polystable if it is the direct sum of stable bundles of the same slope.

Definition 7. We say that a connection A is traceless Hermitian-Yang-Mills (w.r.t. ω) if Tr F_A is harmonic and

$$F_A^o \in \Omega^{1,1}_{prim}(\mathfrak{su}_E)\,.$$

Remark 8. We can prove that for a Kähler manifold (M, ω), and $E \to M$ a $\mathrm{U}(r)$-bundle with a connection A, A is ω-HYM $\iff$ A is traceless ω-HYM and $c_1(E) \in H^{1,1}(M)$. However, we shall not need this.

Note that for a traceless ω-HYM connection, the first Chern class $c_1(E) \in H^2(M, \mathbb{Z})$ may be arbitrary. There are not restrictions with respect to the Hodge decomposition.

Lemma 9. *Let (M, ω) be a projective complex manifold, and A a traceless ω-HYM connection. Then β is an algebraic class (i.e. defined by a rational algebraic cycle).*

Proof. Consider the bundle $\mathfrak{su}_E \subset \mathrm{End}\, E$. This has a product connection $\tilde{A} = A \otimes \mathrm{Id} - \mathrm{Id} \otimes A^t$. This connection has zero trace, and it is traceless ω-HYM, since

$$F_{\tilde{A}}(\phi) = [F_A, \phi] \in \Omega^{1,1}_{prim}(\mathfrak{su}_E),$$

for $\phi \in \Gamma(\mathfrak{su}_E)$. So $F_{\tilde{A}} \in \Omega^{1,1}_{prim}(\mathfrak{su}_E)$ (the adjoint bundle of $\mathfrak{su}_E$ is $\mathfrak{su}_E$ again). Hence $\tilde{A}$ is ω-HYM. Therefore $\mathfrak{su}_E$ is a holomorphic vector bundle, which is moreover polystable. Also $c_1(\mathfrak{su}_E) = 0$ and $c_2(\mathfrak{su}_E) = 2rc_2 - (r-1)c_1^2 = 2r\,\beta$. So $\beta(\mathfrak{su}_E) = 2r\,\beta$. Hence β is an algebraic class. $\square$

Proposition 10 ([9])**.** *Let (M, ω) be a projective complex manifold, $H = [\omega]$. Let $\beta_0 \in H^{2,2}(M)$ be an algebraic class (a Hodge class representable by an algebraic cycle). Then there exists $k \in \mathbb{Z}$ and a ω-HYM bundle $E \to M$ with $\beta(E) = \beta_0 + kH^2$.* $\square$

3.3. **Relationship of ω-HYM and Spin(7)-connections.** Now let (M, ω) be a Calabi-Yau manifold, i.e. M has an $\mathrm{SU}(4)$-structure, and therefore an induced $\mathrm{Spin}(7)$-structure. Suppose that $E \to M$ is a complex rank r bundle with a hermitian metric, and let A be a $\mathrm{U}(r)$-connection. The following result appears already in [6] with a slightly different formulation.

Proposition 11. *We have the following equivalent statements:*

(a) *A is a traceless ω-Hermitian-Yang-Mills connection.*
(b) *A is a* Spin(7)*-connection and $\beta \in H^{2,2}(M)$.*
(c) *A is a* Spin(7)*-connection and $\beta \cup [\mathrm{Re}(\theta)] = 0$.*

Proof. First assume (a). Denote $R = F_A^o$. As A is traceless ω-HYM, we have that $R^{2,0} = 0$ and $\langle R, \omega \rangle = 0$. Now the decomposition $\bigwedge^2_7 = \langle \omega \rangle \oplus A_+$ and $A_+ \subset \triangle^{2,0}$, give

us that $\pi_7(R) = 0$, as required. Moreover, $\mathrm{Tr}\,(R \wedge R)$ represents β, up to a factor, and this is of type $(2,2)$. (b) follows.

The implication (b) $\Rightarrow$ (c) is clear.

Finally, assume (c). So $R = F_A^o \in \underline{\bigwedge^2_{21}}$. This means that $\langle R, \omega \rangle = 0$ and $\pi_{A_+}(R^{2,0} + R^{0,2}) = 0$. So $L(R^{2,0}) = -R^{0,2} = -\overline{R^{2,0}}$. This implies that $R^{2,0} \wedge R^{2,0} = -\frac{1}{4}|R^{2,0}|^2\,\theta$ and hence

$$\begin{aligned} 0 = 8\pi^2 \beta \cup [\mathrm{Re}(\theta)] &= \int_M \mathrm{Tr}\,(R \wedge R) \wedge \mathrm{Re}(\theta) \\ &= \int_M \mathrm{Tr}\,(R^{2,0} \wedge R^{2,0} + R^{0,2} \wedge R^{0,2}) \wedge (\theta + \bar\theta)/2 \\ &= -\,2 \int_M \frac{1}{4} |R^{2,0}|^2 \, \frac{\theta \wedge \bar\theta}{2} \\ &= -\,4\,||R^{2,0}||^2\,. \end{aligned}$$

So $R^{2,0} = 0$ and A is traceless ω-Hermitian-Yang-Mills. $\square$

4. Rotation of complex structures on the 8-torus

Now we shall study explicitly the case of the 8-torus. Let $V = \mathbb{R}^8$ be an 8-dimensional real vector space, and let $\Lambda \subset V$ be a lattice. The group $\mathrm{GL}(8,\mathbb{R})$ acts on V. We consider the torus

$$X = V/\Lambda,$$

and let g be a Riemannian flat metric on X. This is the same as a metric on V, i.e. a reduction $\mathrm{SO}(8) \subset \mathrm{GL}(8,\mathbb{R})$. Note that the set of (flat) metrics g is parametrised by $\mathrm{GL}(8,\mathbb{R})/\,\mathrm{SO}(8)$.

A complex structure J on X which makes X into a complex torus is a complex structure $J : V \to V$, $J^2 = -\,\mathrm{Id}$. This is a reduction to $\mathrm{GL}(4,\mathbb{C}) \subset \mathrm{GL}(8,\mathbb{R})$. If we require that J be compatible with a metric g, then we are asking for a reduction to $\mathrm{U}(4) \subset \mathrm{SO}(8)$. Alternatively, if there is a complex structure J, and we want to put a Kähler metric on X, we look for subgroups $G \cong \mathrm{SO}(8)$ containing $\mathrm{U}(4)$. This is parametrized by an open cone in $\mathrm{Re}(\bigwedge^{1,1}) \cong \mathbb{R}^{16}$. This set is $\mathrm{GL}(4,\mathbb{C})/\,\mathrm{U}(4)$.

If we have a complex structure J with a Kähler metric ω, and we want to get an $\mathrm{SU}(4)$-structure, then we have to fix a generator $\theta \in \bigwedge^{4,0}$ with $|\theta| = 4$. This is parametrized by $\mathrm{U}(4)/\,\mathrm{SU}(4) = S(\bigwedge^{4,0})$.

Now suppose that we have an $\mathrm{SU}(4)$-structure (J, ω, g, θ). If we conjugate by an element $\phi \in \mathrm{Spin}(7)$, then we get a new $\mathrm{SU}(4)$-structure $(J', \omega', g, \theta') = \phi_*(J, \omega, g, \theta)$, where

$$\Omega = \frac{1}{2}\omega^2 + \mathrm{Re}(\theta) = \frac{1}{2}(\omega')^2 + \mathrm{Re}(\theta').$$

By Lemma 1, the possible Kähler forms are elements of $\bigwedge^2_7 = \langle\omega\rangle \oplus A_+$ of norm 2. Therefore

$$\omega' = 2\,\frac{\omega+\gamma}{|\omega+\gamma|}$$

and $\gamma \in A_+$ (note that the metric is fixed in this process, so $|\cdot|$ has a clear meaning). Note that $\gamma = \frac{1}{2}(c+\bar{c})$, $c \in \bigwedge^{2,0}_\omega$. The condition $\gamma \in A_+$ is rewritten as $L(c) = \bar{c}$, which in turn can be rewritten as $c \wedge c = |c|^2\,\frac{\theta}{4}$. There is a possible θ satisfying this if and only if $|c \wedge c| = |c|^2$.

Proposition 12. *Let X be an 8-torus with an* SU(4)*-structure. Let $E \to X$ be a* U(r)*-bundle with $\beta = \beta(E) \in H^4(X)$ and $A \in \mathcal{A}_E$. Take $\gamma \in A_+$, $\gamma = \frac{1}{2}(c+\bar{c})$, $c \in \bigwedge^{2,0}_\omega$, and $\omega' = 2(\omega+\gamma)/|\omega+\gamma|$.*

Suppose $\beta \in \bigwedge^{2,2}_\omega$. Then A is traceless ω'-HYM if and only if the following conditions are satisfied:

- *A is* Spin(7)*-instanton,*
- *$\beta \wedge c \wedge \bar{c} = \frac{1}{4}|c|^2(\beta \wedge \omega^2)$, where $\frac{1}{2}\omega^2 \wedge c \wedge \bar{c} = |c|^2\mathrm{vol}$.*

Proof. By Proposition 11, A is traceless ω'-HYM $\iff$ A is a Spin(7)-instanton and $\beta \cup [\mathrm{Re}(\theta')] = 0$. So we need to see that this latter condition is satisfied.

As β is of type $(2,2)$ w.r.t. ω, we have that $\beta \cup [\mathrm{Re}(\theta)] = 0$. The equality $\beta \wedge \mathrm{Re}(\theta) = \beta \wedge \mathrm{Re}(\theta') = 0$ is equivalent to

$$\beta \wedge \omega^2 = \beta \wedge (\omega')^2\,,$$

since $\Omega = \frac{1}{2}\omega^2 + \mathrm{Re}(\theta) = \frac{1}{2}(\omega')^2 + \mathrm{Re}(\theta')$ is fixed. We rewrite this as

$$\frac{1}{4}|\omega+\gamma|^2(\beta \wedge \omega^2) = \beta \wedge (\omega+\gamma)^2\,.$$

Now $|\omega+\gamma|^2 = |\omega|^2 + |\gamma|^2 = 4 + |\gamma|^2$, since $\bigwedge^2_7 = \langle\omega\rangle \oplus A_+$ is orthogonal. So we get

$$\frac{1}{4}|\gamma|^2(\beta \wedge \omega^2) = 2\beta \wedge \omega \wedge \gamma + \beta \wedge \gamma^2 = \beta \wedge \gamma^2\,,$$

since $\beta \wedge \omega \wedge \gamma = 0$, because β is of type $(2,2)$ (w.r.t. ω).

Now using that $\gamma = \frac{1}{2}(c+\bar{c})$, we rewrite this as

$$\frac{1}{4}|c|^2(\beta \wedge \omega^2) = \beta \wedge c \wedge \bar{c}\,,$$

as required. □

Theorem 13. *Let X be an 8-torus with an* SU(4)*-structure. Take $\gamma \in A_+$, $\gamma = \frac{1}{2}(c+\bar{c})$, and $\omega' = 2(\omega+\gamma)/|\omega+\gamma|$. Let $\beta \in H^4(X,\mathbb{Z})$. If*

- *β is traceless ω-HYM.*
- *$\beta \wedge c \wedge \bar{c} = 6k\,|c|^2\mathrm{vol}$, where $\beta \wedge \omega^2 = 24k\,\mathrm{vol}$ and $\omega^2 \wedge c \wedge \bar{c} = 2|c|^2\mathrm{vol}$.*

Then β is traceless ω'-HYM.

Proof. If β is ω-HYM, then it is a Spin(7)-connection and of type $(2,2)$ by Proposition 11. Now $\beta\wedge c\wedge\bar c = 6k\,|c|^2\mathrm{vol} = \frac14|c|^2(\beta\wedge\omega^2)$. Proposition 12 gives the result. □

The equality defined in the second line of Theorem 13 is rewritten as

$$(\beta - 3k\,\omega^2)\wedge\gamma^2 = 0\,, \tag{6}$$

where

$$\beta = \beta_0 + \omega\wedge\beta_1 + k\,\omega^2,$$

β_0, β_1 are primitive $(2,2)$ and $(1,1)$-forms, respectively.

Remark 14. Equation (6) has a natural symmetry when we *rotate back* from ω' to ω. First, we compute the value k' of β with respect to ω'.

$$\alpha\wedge(\omega')^2 = \alpha\wedge\left(2\frac{\omega+\gamma}{|\omega+\gamma|}\right)^2 = 4\alpha\wedge\frac{\omega^2+2\gamma\wedge\omega+\gamma^2}{4+|\gamma|^2} = 4\frac{24k+6k|\gamma|^2}{4+|\gamma|^2}\mathrm{vol} = 24k\,\mathrm{vol},$$

and

$$(\omega')^4 = \frac{16(\omega+\gamma)^4}{|\omega+\gamma|^4} = \frac{16(24+6\cdot 2|\gamma|^2+\frac32|\gamma|^4)}{(4+|\gamma|^2)^2}\mathrm{vol} = \frac{24\cdot 16(1+\frac14|\gamma|^2)^2}{(4+|\gamma|^2)^2}\mathrm{vol} = 24\mathrm{vol},$$

where we have used that

$$\gamma^4 = \frac{1}{16}(c+\bar c)^4 = \frac{6}{16}c^2\wedge\bar c^2 = \frac{6}{16}|c|^4\frac{1}{16}\theta\wedge\bar\theta = \frac32|\gamma|^4\mathrm{vol}.$$

So $k'=k$. Now to rotate back, we have to take

$$\gamma' = -\frac{2}{|\omega+\gamma|}\left(\gamma - \frac14|\gamma|^2\omega\right)\in A'_+$$

(note that $|\gamma'|=|\gamma|$ and $\langle\omega',\gamma'\rangle=0$), so that

$$\omega = 2\frac{\omega'+\gamma'}{|\omega'+\gamma'|}\,.$$

We compute

$$
\begin{aligned}
(\alpha - 3k\,(\omega')^2) \wedge (\gamma')^2 &= (\alpha - 3k\, \left(2\frac{\omega+\gamma}{|\omega+\gamma|}\right)^2) \wedge \frac{4}{|\omega+\gamma|^2}\left(\gamma - \frac{1}{4}|\gamma|^2\omega\right)^2 \\
&= \frac{4}{|\omega+\gamma|^2}\left(\alpha \wedge \left(\gamma - \frac{1}{4}|\gamma|^2\omega\right)^2 - 12k\,\left(\frac{\omega+\gamma}{|\omega+\gamma|}\right)^2 \wedge \left(\gamma - \frac{1}{4}|\gamma|^2\omega\right)^2\right) \\
&= \frac{4}{|\omega+\gamma|^2}\left(\left(6k|\gamma|^2 + \frac{1}{16}|\gamma|^4 24k\right) - \frac{12k}{|\omega+\gamma|^2}\left(\frac{1}{16}|\gamma|^4\omega^4 + \gamma^4 + (1 - |\gamma|^2 + \frac{1}{16}|\gamma|^4)\omega^2\wedge\gamma^2\right)\right) \\
&= \frac{4}{4+|\gamma|^2}\left(\frac{6}{4}k|\gamma|^2(4+|\gamma|^2) - \frac{12k}{4+|\gamma|^2}\left(\frac{24}{16}|\gamma|^4 + \frac{3}{2}|\gamma|^4 + 2|\gamma|^2(1 - |\gamma|^2 + \frac{1}{16}|\gamma|^4)\right)\right) \\
&= \frac{4}{4+|\gamma|^2}\left(\frac{6}{4}k|\gamma|^2(4+|\gamma|^2) - \frac{24k}{4+|\gamma|^2}(1 + \frac{1}{4}|\gamma|^2)^2|\gamma|^2\right) = 0\,,
\end{aligned}
$$

so equation (6) is satisfied w.r.t. ω', as expected.

We shall need the following later.

Definition 15. We say that $\beta \in H^4(X,\mathbb{Q})$ is asymptotically traceless ω-HYM if there is some $N \gg 0$ such that $N\,\beta$ is traceless ω-HYM.

A class β that is asymptotically traceless ω-HYM in a projective manifold X is a (rational) algebraic cycle.

Let $X = \mathbb{C}^4/\Lambda$, $X' = \mathbb{C}^4/\Lambda'$ be isogenous complex tori, then the asymptotically traceless HYM cycles are the same. We may suppose that $\pi : X' \to X$ is a finite covering, corresponding to an inclusion $\Lambda' \subset \Lambda \subset V = \mathbb{C}^4$. There is a natural isomorphism $\pi^* : H^*(X,\mathbb{Q}) \xrightarrow{\cong} H^*(X',\mathbb{Q})$.

Lemma 16. *Fix Kähler forms ω on X and $\omega' = \pi^*\omega$ on X'. Under π^*, the asymptotically traceless HYM classes on X and on X' are the same.*

Proof. If $\beta \in H^4(X)$ is traceless ω-HYM, then $\pi^*\beta \in H^4(X')$ is ω'-HYM. Just pullback the bundle and connection under $\pi : X' \to X$.

If $\beta' \in H^4(X')$ is ω'-HYM. Let E' be the corresponding holomorphic polystable bundle. Assume for the moment that E' is stable. Consider $E = (\pi_*E')^G$, where $G = \Lambda/\Lambda'$, which is a finite group. Then E is a holomorphic bundle such that $\pi^*E \cong E'$. E is stable with respect to ω: consider a subsheaf $F \subset E$, then $\pi^*F \subset E'$, hence $\deg_{\omega'}\pi^*F < 0$ and hence $\deg_\omega F < 0$. Finally, if E' is a direct sum of stable bundles, then the same holds for E. ∎

5. A Bogomolov type inequality for complex 4-tori

Let us consider a Kähler 4-torus $X = \mathbb{C}^4/\Lambda$ with Kähler form $\omega \in \bigwedge^{1,1}$. Let $E \to X$ be an ω-polystable holomorphic bundle. The celebrated Bogomolov inequality says that [1]

$$\left(c_2 - \frac{r-1}{2r} c_1^2\right) \cup [\omega]^2 \geq 0. \tag{7}$$

We want to strengthen this inequality by using the theory of Spin(7)-instantons developed in previous sections.

First of all, note that the (poly)stability condition is not affected by tensoring E with a line bundle L. The invariant

$$\beta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 \in H^4(M, \mathbb{Q})$$

satisfies that $\beta(E \otimes L) = \beta(E)$ for any line bundle L (actually it is the only such invariant of degree 4 up to a scalar multiple).

As E is a ω-polystable bundle, it admits a ω-HYM connection A by Proposition 6. Note that $\beta \in \bigwedge^{2,2}$. We decompose

$$\beta = \beta_0 + \beta_1 \wedge \omega + k\, \omega^2 \,,$$

where $\beta_0 \in \bigwedge^{2,2}_{prim}$ and $\beta_1 \in \bigwedge^{1,1}_{prim}$. Bogomolov inequality (7) says that

$$k \geq 0.$$

Moreover, if $k = 0$ then $\beta = 0$ (see [1]).

Proposition 17. *Fix an* SU(4)*-structure on* X*, by fixing* $\theta \in \bigwedge^{4,0}$*. Then we have* $(\beta - 3k\, \omega^2) \wedge \gamma^2 \leq 0$*, for all* $\gamma \in A_+$*, or* $(\beta - 3k\, \omega^2) \wedge \gamma^2 \geq 0$*, for all* $\gamma \in A_+$*.*

Proof. First note that A is a Spin(7)-instanton. Consider the Spin(7)-rotation equation

$$(\beta - 3k\, \omega^2) \wedge \gamma \wedge \gamma = 0, \tag{8}$$

for $\gamma \in A_+$, and let $\mathcal{Q}_\omega \subset A_+$ be the space of solutions to (8). This is a (real) quadric in $A_+ \cong \mathbb{R}^6$, where the point $\gamma = 0$ (corresponding to ω) is a singular point. The set of solutions to (8) is reduced to the origin if and only if the quadratic form is definite (positive or negative). Let $\widehat{\mathcal{Q}}_\omega$ be the closure of

$$\left\{ 2\frac{\omega + \gamma}{|\omega + \gamma|} \,|\, \gamma \in \mathcal{Q}_\omega \right\} \subset S(\textstyle\bigwedge^2_7).$$

Suppose that $\gamma \neq 0$ is a solution to (8). Then $\omega' = 2(\omega + \gamma)/|\omega + \gamma|$ gives another complex structure for X for which β is traceless ω'-HYM, by Proposition 12. The set

of solutions to the Spin(7)-rotation equation

$$(\beta - 3k'\,(\omega')^2) \wedge \gamma \wedge \gamma = 0,$$

for $\gamma \in A'_+$ should give the same set of complex structures as (8), that is $\widehat{\mathcal{Q}}_\omega = \widehat{\mathcal{Q}}_{\omega'}$. In particular ω' should be a singular point of this set. The conclusion is that all points in $\mathcal{Q}_\omega$ are singular, i.e., that $\mathcal{Q}_\omega$ is a linear subspace. Equivalently, the defining equation (8) is semi-definite. □

It is of interest to determine the set of (rational) classes which are represented as $\beta(E)$ for polystable bundles E. We define

$$\mathcal{H}_\omega = \{\beta \in H^{2,2}(X) \,|\, \beta \text{ is asymptotically } \omega\text{-HYM}\}.$$

By Lemma 16, if $\pi : X \to X'$ is an isogeny between 4-tori, and $\omega = \pi^*\omega'$, then $\mathcal{H}_\omega = \pi^*\mathcal{H}_{\omega'}$.

Lemma 18. *For any Kähler torus X, the set $\mathcal{H}_\omega$ is a convex cone.*

Proof. It is enough to see that if β_1, β_2 are ω-HYM, then $\beta_1 + \beta_2$ is ω-HYM. Just take bundles $E_1, E_2 \to X$ which are polystable, and with trivial determinant, with $\beta(E_i) = \beta_i$, $i = 1, 2$. Then $E_1 \oplus E_2$ is polystable and $\beta(E) = \beta_1 + \beta_2$. □

Suppose from now on that (X, ω) is an abelian variety.

Proposition 19. *If X is an abelian variety then*

$$(\beta - 3k\,\omega^2) \wedge \gamma \wedge \gamma \leq 0,$$

for all $\gamma \in A_+$. If there is equality, then β is Spin*-rotable via γ.*

Proof. We have to see that (8) is *negative* semi-definite. For $\beta \in \mathcal{H}_\omega$, consider the quadratic form on A_+,

$$\Phi_\beta(\gamma, \gamma) = (\beta - 3k\,\omega^2) \wedge \gamma \wedge \gamma\,.$$

Let $\beta, \beta' \in \mathcal{H}_\omega$. Then $t\beta + (1-t)\beta' \in \mathcal{H}_\omega$ for $t \in [0,1] \cap \mathbb{Q}$. The form

$$\Phi_{t\beta+(1-t)\beta'} = t\Phi_\beta + (1-t)\Phi_{\beta'}$$

is semidefinite for all $t \in [0,1] \cap \mathbb{Q}$. This implies that either Φ_β and $\Phi_{\beta'}$ are semidefinite of the same type (positive or negative) or that there is some $\lambda \in \mathbb{R}$ with $\Phi_\beta = \lambda \Phi_{\beta'}$. By (3), the set $\gamma \wedge \gamma$ expands $\bigwedge^{2,2}_{prim}$. So in the second case it must be $\beta = \lambda\beta'$. Therefore λ must be rational and positve, and Φ_β, $\Phi_{\beta'}$ are semidefinite of the same type.

Now the proposition follows in the case of X being projective by noting that ω^2 is asymptotically ω-HYM (by Proposition 10) and $\Phi_{\omega^2} = -2\omega^2 < 0$. □

Remark 20. Proposition 19 also holds if X is a complex torus with $H^{1,1}_{prim}(X) \cap H^2(X,\mathbb{Q}) \neq 0$. Take a non-zero $\alpha \in H^{1,1}_{prim}(X) \cap H^2(X,\mathbb{Q})$, and let $\mathcal{L}$ be a line bundle with $c_1(\mathcal{L}) = \alpha$. Then $E = \mathcal{L} \oplus \mathcal{L}^{-1}$ is a polystable rank 2 bundle with $\beta = -\alpha^2$. By Lemma 24, $\Phi_{-\alpha^2}$ is negative semidefinite.

We conjecture that 19 is true for any Kähler complex 4-torus.

Theorem 21. *Consider $\beta_0 \in \bigwedge^{2,2}_{prim}$. Let*

$$k_m = k_m(\beta_0) = \frac{1}{4} \min\{\beta_0 \wedge c \wedge \bar{c} \,|\, c \in \bigwedge\nolimits^{2,0},\, |c| = 1\},$$

and

$$\mathcal{R} = \{c \in \bigwedge\nolimits^{2,0} \,|\, \beta_0 \wedge c \wedge \bar{c} = k_m,\, |c \wedge c| = 1,\, |c| = 1\}.$$

Then for any $\beta = \beta_0 + \beta_1 \wedge \omega + k\,\omega^2$ which is a traceless ω-HYM class, we have $k \geq k_m$. And $k = k_m$ if and only if β is Spin*-rotable (via any $\gamma = \frac{1}{2}(c + \bar{c})$, $c \in \mathcal{R}$).*

Proof. Let $\beta = \beta_0 + \beta_1 \wedge \omega + k\,\omega^2$ be an ω-HYM class. By (3), $\beta_1 \wedge \omega \wedge \gamma \wedge \gamma = 0$, for any $\gamma \in A_+$. By Proposition 19, $(\beta - 3k\,\omega^2) \wedge \gamma \wedge \gamma \leq 0$. Therefore $(\beta_0 - 2k\,\omega^2) \wedge \gamma \wedge \gamma \leq 0$, for $\gamma \in A_+$. Write $\gamma = \frac{1}{2}(c + \bar{c})$. Then $\gamma' = \frac{1}{2}(ic - i\bar{c}) \in A_-$ and $(\beta_0 - 2k\,\omega^2) \wedge (\gamma')^2 = (\beta_0 - 2k\,\omega^2) \wedge \gamma^2 \leq 0$, so the inequality holds on A_-. Now take a general $\gamma \in \triangle^{2,0}$, $\gamma = \gamma_+ + \gamma_-$, with $\gamma_\pm \in A_\pm$. Then $\beta_0 \wedge \gamma_+ \wedge \gamma_- = 0$ by (4) and $\omega^2 \wedge \gamma_+ \wedge \gamma_- = 2\langle \gamma_+, \gamma_- \rangle = 0$. So

$$(\beta_0 - 2k\,\omega^2) \wedge \gamma^2 = (\beta_0 - 2k\,\omega^2) \wedge \gamma_+^2 + (\beta_0 - 2k\,\omega^2) \wedge \gamma_-^2 \leq 0.$$

Take $\gamma \in \triangle^{2,0}$ and write $\gamma = \frac{1}{2}(c + \bar{c})$, $c \in \bigwedge^{2,0}$. Then $(\beta_0 - 2k\,\omega^2) \wedge c \wedge \bar{c} \leq 0$, or equivalently

$$4k_m \leq \beta_0 \wedge c \wedge \bar{c} \leq 2k\,\omega^2 \wedge c \wedge \bar{c} = 4k\,|c|^2 = 4k.$$

If $k = k_m$, then β is Spin-rotable for any $\gamma \in A_+$ satisfying equality. But $\gamma = \frac{1}{2}(c+\bar{c})$ is in A_+ when $c \wedge c = |c|^2 \frac{\theta}{4}$. This happens for a suitable θ if and only if $|c \wedge c| = |c|^2$. $\square$

Let $\mathcal{D}^2 \subset H^{2,2}(X)$ be the set of Hodge $(2,2)$-classes represented by algebraic cycles (all of them if we assume the Hodge Conjecture). Decompose orthogonally $\mathcal{D}^2 = \mathcal{D}^2_o \oplus \mathbb{Q}\omega^2$. Then there is a convex radial function $f : \mathcal{D}^2_o \to [0,\infty)$ (i.e., $f(t\beta) = t \cdot f(\beta)$ for $t > 0$) such that the closure of $\mathcal{H}_\omega$ is

$$\mathrm{cl}(\mathcal{H}_\omega) = \{\beta + k\,\omega^2 \,|\, \beta \in \mathcal{D}^2_o,\, k \geq f(\omega)\}.$$

This is a consequence of Proposition 10: for any $\beta \in \mathcal{D}^2_0$, there is some large $k \gg 0$ for which $\beta + k\,\omega^2$ is ω-HYM.

The usual Bogomolov inequality says that $f \geq 0$. Theorem 21 says that

$$f(\beta_0 + \beta_1 \wedge \omega) \geq \max\{k_m(\beta_0), 0\}.$$

Note that $k_m(\beta_0)$ may be negative, in which case there is no improvement. However, if $k_m(\beta_0) = 0$ then $k_m(-\beta_0) > 0$, so $f \neq 0$.

In [18], C. Voisin finds Kähler non-projective 4-tori for which the Hodge Conjecture fails in the following extended version: X has a Hodge $(2,2)$-class not represented by Chern classes of sheaves. Such examples do not have stable bundles because all Hodge $(2,2)$-classes have $k = 0$. It would be interesting to know if the condition $k \geq k_m$ is sufficient for a Hodge $(2,2)$-class to be represented as $\beta(E)$ of a polystable bundle, when $k_m > 0$.

Remark 22. Using the first variation of equation (6), we can prove further: if β is ω-HYM and Spin-rotable via $\gamma \in A_+$, then $(\beta_0 - 2k\,\omega^2) \wedge \gamma = 0$ and $\beta_1 \wedge \omega = \gamma \wedge \gamma'$ for some $\gamma' \in A_-$.

We can rephrase Theorem 21 also in terms of Spin(7)-geometry.

Theorem 23. *Suppose that β is traceless ω-HYM. Then*

$$k = \frac{1}{24} \max\{\beta \wedge (\omega')^2 \,|\, \omega' \in S(\textstyle\bigwedge^2_7)\},$$

and those ω' where this minimum is achieved are exactly the Kähler forms such that β is traceless ω'-HYM.

Proof. Let $\omega' \in S(\bigwedge^2_7)$. Then $\omega' = 2\frac{\omega+\gamma}{|\omega+\gamma|}$, where $\gamma \in A_+$. By Proposition 19, $\beta \wedge \gamma^2 \leq 3k\,\omega^2 \wedge \gamma^2 = 6k\,|\gamma|^2$. Then

$$\begin{aligned} \beta \wedge (\omega')^2 &= \beta \wedge 4\frac{(\omega+\gamma)^2}{|\omega+\gamma|^2} = \frac{4}{4+|\gamma|^2}\,\beta \wedge (\omega^2 + 2\omega \wedge \gamma + \gamma^2) \\ &\leq \frac{4}{4+|\gamma|^2}(24k + 6k|\gamma|^2)\mathrm{vol} = 24\,k\,\mathrm{vol}. \end{aligned}$$

We defined k' by $\beta \wedge (\omega')^2 = 24\,k'\mathrm{vol}$. Therefore $k' \leq k$. If we have equality, then β is Spin-rotable via γ. □

6. Examples

We want to see now some examples in which the new Bogomolov inequality

$$(\beta - 3k\omega^2) \wedge c \wedge \bar{c} \leq 0 \tag{9}$$

is satisfied, and examples where it puts new constraints for the existence of stable bundles.

6.1. **Complete intersections.** A basic case is that of degree 4 classes β which are complete intersections, i.e., product of divisors (rational degree 2 classes). That is, bundles $E = \mathcal{O}(D_1) \oplus \ldots \oplus \mathcal{O}(D_r)$, where $D_1 \cdot H = \ldots = D_r \cdot H$, where H denotes the polarization. This is case where E is H-polystable and completely decomposable.

We start with a useful lemma.

Lemma 24. *Let $\alpha \in \triangle^{1,1}_{prim}$. Then $\beta = -\alpha^2$ satisfies (9).*

Proof. Taking suitable coordinates, we can take ω to be standard and $c = dz_{12} + dz_{34}$ (see Remark 4). Write $\alpha = \sum a_{ij} dz_{i\bar{j}}$ and recall that $a_{ji} = -\bar{a}_{ij}$ since α is a real form. In particular a_{ii} are purely imaginary. Denote $a = (a_{ij})$. As α is primitive, we have that $\sum a_{ii} = 0$. Then $0 = (\sum a_{ii})^2 = \sum_{i\neq j} a_{ii}a_{jj} + \sum a_{ii}^2$, hence $\sum_{i\neq j} a_{ii}a_{jj} = \sum |a_{ii}|^2$. Now we compute

$$\begin{aligned} 24k = \beta \wedge \omega^2 &= -\alpha^2 \wedge \omega^2 \\ &= -\sum_{a<b} a_{ij}a_{i'j'} dz_{i\bar{j}i'\bar{j}'} \left(\frac{i}{2}\right)^2 2dz_{a\bar{a}b\bar{b}} \\ &= 16\,\frac{1}{2}\left(-\sum_{i\neq j} a_{ij}a_{ji} + \sum_{i\neq j} a_{ii}a_{jj}\right) \\ &= 8\left(\sum_{i\neq j} |a_{ij}|^2 + \sum_{i} |a_{ii}|^2\right) \\ &= 8\sum |a_{ij}|^2 = 8||a||^2. \end{aligned}$$

An easy computation gives

$$\beta \wedge c \wedge \bar{c} = 16(a_{13}a_{24} + a_{31}a_{42} + a_{14}a_{23} + a_{41}a_{32} + a_{34}a_{43} + a_{12}a_{21} + a_{11}a_{22} + a_{33}a_{44}).$$

Using that $|a_{13}a_{24}| \leq \frac{1}{2}(|a_{13}|^2 + |a_{24}|^2)$, etc, we have that

$$\beta \wedge c \wedge \bar{c} \leq 8||a||^2 = 24\,k = 3k\,|c|^2\,,$$

since $|c| = 2\sqrt{2}$. This is the required inequality. $\square$

Suposse that $E = E_1 \oplus E_2$, where E_1, E_2 are holomorphic bundles of ranks r_1, r_2, and of the same ω-slope. Then

$$\beta(E) = \beta(E_1) + \beta(E_2) - \frac{r_1 r_2}{2(r_1+r_2)}\left(\frac{c_1(E_1)}{r_1} - \frac{c_1(E_2)}{r_2}\right)^2.$$

If E_1, E_2 have the same ω-slope then

$$\alpha = \frac{c_1(E_1)}{r_1} - \frac{c_1(E_2)}{r_2} \in \bigwedge\nolimits^{1,1}_{prim}$$

Hence if $\beta(E_1)$ and $\beta(E_2)$ satisfy (9), then $\beta(E)$ satisfies (9), since $-\alpha^2$ does by Lemma 9 and the sum of classes which satisfy (9) also satisfy it.

Corollary 25. *Let $E = \mathcal{O}(D_1) \oplus \ldots \oplus \mathcal{O}(D_r)$ be a rank r polystable bundle which is a direct sum of line bundles of the same ω-slope. Then $\beta(E)$ satisfies (9).*

6.2. **Diagonal property for self-products.** Consider the Jacobian J of a (generic) genus 2 curve, so J is a 2-dimensional abelian variety which is principally polarised by some divisor θ. By [2], there is a vector bundle $F \to J$ with $c_1 = \theta$, $c_2 = p_0$, the class of a point. In particular, $\beta(F) = c_2 - \frac{1}{4}c_1^2 = \frac{3}{4}p_0$.

Consider the bundle $E \to X = J \times J$, which is the pull-back of F under the addition map $\sigma : J \times J \to J$, $\sigma(x_1, x_2) = x_1 - x_2$. It has $\beta(F) = \frac{3}{4}\Delta$, where Δ is the diagonal. Take the Kähler class $\omega = \sqrt{2}(\theta_1 + \theta_2)$, where $\theta_i = pr_i^*\theta$. Here we have normalized so that $\omega^4 = 24$. So

$$k = \frac{1}{24}\beta \wedge \omega^2 = \frac{1}{32}\Delta \wedge \omega^2 = \frac{1}{2}\,,$$

since $\Delta \wedge \omega^2 = (2\sqrt{2}\theta)^2 = 8$. If we write, in a suitable basis, $\omega = \frac{i}{2}\sum dz_{j\bar{j}}$, where (z_1, z_2) are the coordinates of the first factor J, and (z_3, z_4) are the coordinates of the second factor J, then we can take

$$c_1 = dz_{12} + dz_{34}.$$

Clearly

$$\begin{aligned} &\beta \wedge c_1 \wedge \bar{c}_1 = \frac{3}{4}\Delta \wedge c_1 \wedge \bar{c}_1 = \frac{3}{4}(2dz_{12}) \wedge (2d\bar{z}_{12}) = \frac{3}{4}\,16 = 12, \\ &\omega^2 \wedge c_1 \wedge \bar{c}_1 = 16. \end{aligned}$$

So we have $(\beta - 3k\omega^2) \wedge c_1 \wedge \bar{c}_1 = 0$, which means that β is Spin-rotable. Using the basis in Remark 3, we check that $\beta \wedge c_j \wedge \bar{c}_j \leq 12$, for $j = 2, \ldots 6$, agreeing with the Bogomolov inequality.

Note that the class $D = \Delta - \epsilon\,\omega^2$, $\epsilon > 0$ small, does not satisfy (9), so it cannot be $\beta(E)$ for any polystable bundle E. Even for any large $\ell > 0$, ℓD also cannot be $\beta(E)$ for a polystable bundle E, although the associated value $k = \frac{1}{24}\ell D \cup [\omega]^2$ is as large as we want.

The above situation is isomorphic to taking $J \times \{p_0\} \subset J \times J$. More generally, we can consider $X = Y \times Y'$, where Y, Y' are complex 2-tori, and $F = Y \times \{p_0\}$. Then the class

$$\beta = dz_{12\bar{1}\bar{2}}$$

is Spin-rotable via

$$c \in \langle dz_{12} + dz_{34}, i\,dz_{12} - i\,dz_{34}\rangle\,.$$

We shall see in Subsection 7.1 what is the rotated 4-torus.

6.3. **Weil classes in Weil complex tori.** Weil complex tori are complex tori whose endomorphism ring contains a purely imaginary quadratic field, $\mathbb{Q}[\sqrt{-d}] \subset \mathrm{End}\,(X)$, $d > 0$ square-free. Let us describe them explicitly. Consider the ring $L = \mathbb{Z}[\sqrt{-d}]$ and the field $K = \mathbb{Q}[\sqrt{-d}]$. There is a natural map $\varphi : L \to L$, $\varphi(x) = \sqrt{-d}\, x$.

Consider the lattice $\Lambda = L^4$, and let $V = \Lambda \otimes \mathbb{R} \cong \mathbb{R}^8$. The map $I(x) = \frac{1}{\sqrt{d}}\, \varphi(x)$, $I : V \to V$, defines a complex structure. So $(V, I) \cong (\mathbb{C}^4, i)$. Now choose two I-complex subspaces of dimension 2,

$$V = W_+ \oplus W_- \,,$$

and define

$$J : V \to V, \qquad J|_{W_+} = I, \quad J|_{W_-} = -I\,.$$

Then (V, J) is a complex vector space, $\Lambda \subset V$, and let

$$X = (V, J)/\Lambda\,.$$

This is a complex 4-torus. Note that $IJ = JI$, so $\varphi : X \to X$ is holomorphic, and $\varphi^2 = -d\,\mathrm{Id}$. Hence $\mathbb{Z}[\sqrt{-d}] \subset \mathrm{End}\,(X)$. This is called a *Weil complex torus*. For generic choice of $W_\pm$, we have $\mathbb{Z}[\sqrt{-d}] = \mathrm{End}\,(X)$.

The Weil classes are the $(2, 2)$-classes lying in

$$\begin{aligned} K \cong \bigwedge\nolimits_K^4 (\Lambda \otimes \mathbb{Q}) \subset \bigwedge\nolimits_I^{4,0} V = \bigwedge\nolimits_I^{4,0} (W_+ \oplus W_-) = \\ = \bigwedge\nolimits_I^{2,0} W_+ \otimes \bigwedge\nolimits_I^{2,0} W_- = \bigwedge\nolimits_J^{2,0} W_+ \otimes \bigwedge\nolimits_J^{0,2} W_- \subset \bigwedge\nolimits_J^{2,2} V\,. \end{aligned}$$

These are Hodge classes (rational classes of pure type). We note that the space of Weil classes is a rational vector space of dimension 2. The generic Weil complex torus does not have more Hodge classes [10].

We may put many Kähler forms on X. Note that

$$\bigwedge\nolimits_I^{1,1} V = \bigwedge\nolimits^{1,1} W_+ \oplus \bigwedge\nolimits^{1,1} W_- \ \oplus \bigwedge\nolimits^{1,0} W_+ \bigwedge\nolimits^{0,1} W_- \oplus \bigwedge\nolimits^{0,1} W_+ \bigwedge\nolimits^{1,0} W_-$$

If we consider Kähler forms so that $W_+ \perp W_-$, then ω lies in

$$\bigwedge\nolimits^{1,1} W_+ \oplus \bigwedge\nolimits^{1,1} W_- \subset \bigwedge\nolimits_I^{1,1} V\,. \tag{10}$$

We want the Riemannian metric g to be positive-definite for X, i.e. $g = g_+ + g_-$, where $g_\pm$ is positive definite on $W_\pm$. As $g(x, Jy) = \omega(x, y)$, and $J_\pm = \pm I$ on $W_\pm$, we see that ω defines an *indefinite* semi-riemannian metric $\tilde{g} = g_+ - g_-$ for (V, I). Said otherwise, we have complex coordinates (w_1, w_2, w_3, w_4) for (V, I) such that $\tilde{h} = \frac{i}{2}(dw_{1\bar{1}} + dw_{2\bar{2}} - dw_{3\bar{3}} - dw_{4\bar{4}})$ is an indefinite Kähler form, $\tilde{h} = \tilde{g} + i\omega$, W_+ has coordinates (w_1, w_2) and W_- has coordinates (w_3, w_4). The complex coordinates for (V, J) are $(z_1, z_2, z_3, z_4) = (w_1, w_2, \bar{w}_3, \bar{w}_4)$ and the induced Kähler form is $h = \frac{i}{2}(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}} + dz_{4\bar{4}})$.

Take now a Weil class $\beta_0 = 2\,\mathrm{Re}(dw_{1234}) \in \bigwedge_K^4(K^4)$. In coordinates $(z_1, z_2, z_3, z_4) = (w_1, w_2, \bar{w}_3, \bar{w}_4)$, $\beta_0 = 2\,\mathrm{Re}(dz_{12\bar{3}\bar{4}}) = dz_{12\bar{3}\bar{4}} + dz_{\bar{1}\bar{2}34}$, which is rational and of J-type $(2,2)$, up to normalization (that we shall ignore). Recall that $W_+ = \langle z_1, z_2\rangle$, $W_- = \langle z_3, z_4\rangle$. Using the basis of A_+ in Remark 3, one checks easily that

$$k_m(\beta_0) = 1. \tag{11}$$

The maximum value of $\beta_0 \wedge c \wedge \bar{c}$ is 32 and it is achieved for $c = c_1 = dz_{12} + dz_{34}$.

So if we consider

$$\begin{aligned} \beta_0 &= dz_{12\bar{3}\bar{4}} + dz_{\bar{1}\bar{2}34}, \\ \omega &= \frac{i}{2}(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}} + dz_{4\bar{4}}), \\ \beta &= \beta_0 + k\,\omega^2, \\ k &= 1, \end{aligned} \tag{12}$$

the Spin-rotation equation

$$(\beta_0 - 2k\,\omega^2) \wedge c \wedge \bar{c} = 0$$

has solution

$$c = dz_{12} + dz_{34}\,.$$

Summarizing,

$$\beta = (dz_{12\bar{3}\bar{4}} + dz_{\bar{1}\bar{2}34}) + \omega^2 \tag{13}$$

is a Spin-rotable class, where the rotation is given by $c = dz_{12} + dz_{34}$. We shall see later that Spin-rotation produces a modification of the complex structure on $W_+ = \langle z_1, z_2\rangle$ and on $W_- = \langle z_3, z_4\rangle$, keeping them orthogonal.

6.4. **Bogomolov inequality for surfaces in an abelian four-fold.** Let X be an abelian variety of dimension 4, with polarisation H, and let $Z \subset X$ be a surface which is a locally complete intersection. The Bogomolov inequality for stable bundles produces another inequality for Z by using the Serre construction of stable bundles.

Assume, to start with, that X is irreducible (i.e., it has no subtori). Then Z has ample canonical sheaf K_Z. We want to construct an extension

$$0 \to \mathcal{O}^{r-1} \to E \to \mathcal{O}(H) \otimes I_Z \to 0, \tag{14}$$

where I_Z is the ideal sheaf of Z. As $H^1(X, \mathcal{O}(H)) = H^2(X, \mathcal{O}(H)) = 0$, the long exact sequence for Ext-groups produces an isomorphism

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}(H) \otimes I_Z, \mathcal{O}^{r-1}) &\cong \mathrm{Ext}^2(\mathcal{O}(H) \otimes \mathcal{O}_Z, \mathcal{O}^{r-1}) \\ &\cong H^0(\mathcal{E}\mathrm{xt}^2(\mathcal{O}(H) \otimes \mathcal{O}_Z, \mathcal{O}^{r-1})) \\ &\cong H^0(Z, K_Z(-H))^{r-1}, \end{aligned} \tag{15}$$

where we have used that $\mathcal{E}\mathrm{xt}^2(\mathcal{O}_Z, \mathcal{O}) \cong K_Z$ for a locally complete intersection. If $K_Z(-H)$ is ample on Z, then it is globally generated. As Z is of dimension 2, three generic sections give an epimorphism $\mathcal{O}_Z^3 \to K_Z(-H)$. Take $r = 4$ and use the extension class that they define under the isomorphisms (15). This guarantees that the sheaf E defined in (14) is a bundle. If $K_Z(-H)$ is not ample, we work as follows: there is a small rational $\ell = \frac{1}{m} > 0$ such that $K_Z(-\ell H)$ is ample. Take a covering $\pi : X' \to X$ given by the lattice $\Lambda' = m\Lambda$. Change Z by $Z' = \pi^{-1}(Z)$, and use the polarisation $H' = \frac{1}{m}H$ for X'.

If $\operatorname{Pic} X = \mathbb{Z}$, and we take H to be the minimal generator, then E defined in (14) is stable: let $F \subset E$ be a subsheaf, that we may assume it is saturated, then it is written as an extension $F_1 \to F \to F_2$, where $F_1 \subset \mathcal{O}^{r-1}$ and $F_2 \subset \mathcal{O}(H) \otimes I_Z$. This implies that $\deg_H F_1 \leq 0$, and F_2 is either zero or $F_2 = \mathcal{O}(D) \otimes I_W$ with $D \leq H$. If $D < H$, then $\deg_H F_2 \leq 0$. If $D = H$, then the quotient exact sequence is of the form $G_1 \to G \to T$, where T is torsion. If $c_1(F) = H$, then $c_1(G) = 0$, but we have a quotient $\mathcal{O}^{r-1} \to G_2 \subset G$. This implies that $T = 0$, and $G_2 = \mathcal{O}^{r'}$. Then the bundle should split off a summand G_2, which is impossible. Therefore $c_1(F) \leq 0$, and E is H-stable.

If $\operatorname{Pic} X \neq \mathbb{Z}$, then take $D > 0$ an integral divisor with $\deg_H D$ minimal. An stable bundle is constructed now as an extension

$$0 \to \mathcal{O}^{r-1} \to E \to \mathcal{O}(D) \otimes I_Z \to 0, \tag{16}$$

as before, with the difference that now $c_1(E) = D$ and $c_2(E) = Z$. So

$$\beta = \beta(E) = Z - \frac{r-1}{2r} D^2 .$$

Here

$$k = \frac{\beta \cdot H^2}{H^4} = \frac{Z \cdot H^2}{H^4} - \frac{r-1}{2r}\frac{D^2 \cdot H^2}{H^4} .$$

The Bogomolov inequality says that $(\beta - 3kH^2) \wedge c \wedge \bar{c} \leq 0$, so

$$\left(Z - \frac{r-1}{2r} D^2 - 3 \left(\frac{Z \cdot H^2}{H^4} - \frac{r-1}{2r}\frac{D^2 \cdot H^2}{H^4} \right) H^2 \right) \wedge c \wedge \bar{c} \leq 0.$$

Now change X by a cover $X' \to X$ corresponding to the polarisation ℓH, with $\ell = \frac{1}{m}$. Use $H' = \ell H$ and $D' = \ell D$. We get

$$\left(Z - \ell^2 \frac{r-1}{2r} D^2 - 3 \left(\frac{Z \cdot H^2}{H^4} - \ell^2 \frac{r-1}{2r}\frac{D^2 \cdot H^2}{H^4} \right) H^2 \right) \wedge c \wedge \bar{c} \leq 0.$$

The leading term is

$$\left(Z - 3\frac{Z \cdot H^2}{H^4} H^2 \right) \wedge c \wedge \bar{c} \leq 0.$$

Note that $Z \wedge c \wedge \bar{c} = \int_Z |c|^2 = ||c||_Z^2$, $Z \cdot \frac{1}{2} H^2 = \mathrm{vol}(Z)$, $\frac{1}{24} H^4 = \mathrm{vol}(X)$ and $H^2 \wedge c \wedge \bar{c} = 2||c||_X^2$. Thus

$$\frac{||c||_Z^2}{\mathrm{vol}(Z)} \leq \frac{1}{2} \frac{||c||_X^2}{\mathrm{vol}(X)},$$

for any $c \in A_+$.

7. Rotating in 2 dimensions

There is an analogue of the Spin-rotation in 2 complex dimensions, that is worth to review. It is closely related to the twistorial construction. We mention that this idea has been used by M. Toma [17] to produce stable bundles on complex 2-tori.

Let $X = V/\Lambda$ be a 2-dimensional complex torus, with a Kähler form ω, and complex structure J. Let Π be its period matrix, that is the 2×4-matrix whose columns are the vectors of a basis of the lattice Λ, with respect to some complex coordinates of $V \cong \mathbb{C}^2$. This is well defined up to the action of a matrix of $\mathrm{GL}(2, \mathbb{C})$ on the left and a matrix of $\mathrm{GL}(4, \mathbb{Z})$ on the right.

In 2 dimensions, we have the inclusion of groups $U(2) \subset SO(4)$, and the quotient

$$SO(4)/U(2) = S^2 = S(\bigwedge\nolimits_+^2)$$

parametrizes complex structures on X compatible with the metric. Here

$$\bigwedge\nolimits_+^2 = \triangle^{2,0} \oplus \langle \omega \rangle, \qquad \bigwedge\nolimits_-^2 = \triangle^{1,1}_{prim}.$$

We can rotate the complex structure by taking $r, s \in \mathbb{R}$, $r^2 + s^2 = 1$ and considering the new Kähler form

$$\omega' = r\,\omega + s\,\gamma,$$

where $\gamma \in \triangle^{2,0}$, $|\gamma| = |\omega| = \sqrt{2}$. This defines a new complex structure J' by the equation $g(x, y) = \omega'(x, J'y)$.

Let us find explicitly the new abelian variety $X' = (V, J')/\Lambda$. For a suitable choice of complex coordinates (z_1, z_2) for $(V, J) = \mathbb{C}^2$, we may write

$$\omega = \frac{i}{2}(dz_{1\bar{1}} + dz_{2\bar{2}}),$$
$$\gamma = \frac{1}{2}(dz_{12} + dz_{\bar{1}\bar{2}}),$$

where we abbreviate $dz_{12} = dz_1 \wedge dz_2$, $dz_{1\bar{1}} = dz_1 \wedge d\bar{z}_1$, etc.

We introduce real coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, so that

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2,$$
$$\gamma = dx_1 \wedge dx_2 - dy_1 \wedge dy_2.$$

Therefore

$$\omega' = r\,dx_1 \wedge dy_1 + r\,dx_2 \wedge dy_2 + s\,dx_1 \wedge dx_2 - s\,dy_1 \wedge dy_2.$$

Using the equation $g(x,y) = \omega'(x, J'y)$, we easily compute the matrix of J' in this basis:

$$J' = \begin{pmatrix} 0 & -r & -s & 0 \\ r & 0 & 0 & s \\ s & 0 & 0 & -r \\ 0 & -s & r & 0 \end{pmatrix}.$$

We want to put new coordinates so that J' becomes the standard complex structure. These coordinates are given by

$$\begin{pmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{pmatrix} = M \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}, \text{ where } M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & s & 0 \\ 0 & -s & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The new complex coordinates are $z_1' = x_1' + iy_1'$, $z_2' = x_2' + iy_2'$.

Conjugating by M, we see that the new complex torus is $X' \cong \mathbb{C}^2/\Lambda'$, where the matrix period corresponding to $\Lambda' = M\,\Lambda$ is

$$\Pi' = M\,\Pi$$

(writing the matrix Π as a 4×4-real matrix by putting the real and imaginary parts of each complex entry vertically).

7.1. **Rotating a product of two 2-tori.** Let us start by looking at a simple case of Spin-rotation of a complex 4-torus which is the product of two complex 2-tori. Let $X = Y_1 \times Y_2$, where Y_1, Y_2 are complex 2-tori. Put a product metric, and orthonormal complex coordinates (z_1, z_2, z_3, z_4), where (z_1, z_2) are coordinates of Y_1 and (z_3, z_4) are coordinates of Y_2. Consider the class

$$\beta = dz_{12\bar{1}\bar{2}} \in H^4(X).$$

This is the Poincaré dual of a multiple of $F = Y_1 \times \{pt\} \subset X$, which is an algebraic class. Up to a multiple, β is a rational class. By [2], there is a stable rank 2 bundle $E \to Y_2$ with $c_1 = 0$, $c_2 = \{pt\}$. Pulling it back to X, we see that β is ω-HYM (up to a multiple, which we shall ignore).

We compute the value

$$k = \frac{\beta \wedge \omega^2}{\omega^4} = \frac{-\frac{1}{2} dz_{12\bar{1}\bar{2}} \wedge dz_{3\bar{3}4\bar{4}}}{\frac{24}{16} dz_{1234\bar{1}\bar{2}\bar{3}\bar{4}}} = \frac{1}{3}\,.$$

Any rotation parameter

$$c \in \langle dz_{12} + dz_{34}, i\,dz_{12} - i\,dz_{34} \rangle$$

satisfies the rotation equation (6): take $c = \frac{1}{2}(w\,dz_{12} + \bar{w}dz_{34})$, where $w \in \mathbb{C}$. Then $|c| = \sqrt{2}\,|w|$, and we compute

$$(\beta - 3k\,\omega^2) \wedge c \wedge \bar{c} = \frac{1}{4}|w|^2\,16\,\mathrm{vol} - 3k\,2(\sqrt{2}\,|w|)^2\mathrm{vol} = 0.$$

Applying the Spin-rotation, we get a new complex structure J' given by the same Riemannian metric and the Kähler form

$$\omega' = \frac{1}{\sqrt{1+|w|^2}}\left(\frac{i}{2}(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}} + dz_{4\bar{4}}) + \frac{1}{2}(w\,dz_{12} + \bar{w}dz_{\bar{1}\bar{2}} + \bar{w}dz_{34} + w\,dz_{\bar{3}\bar{4}})\right).$$

Clearly, $\omega' = \omega'_1 + \omega'_2$ where $\omega'_1 = \frac{1}{\sqrt{1+|w|^2}}(\frac{i}{2}(dz_{1\bar{1}} + dz_{2\bar{2}}) + \frac{1}{2}(w\,dz_{12} + \bar{w}dz_{\bar{1}\bar{2}}))$ and $\omega'_2 = \frac{1}{\sqrt{1+|w|^2}}(\frac{i}{2}(dz_{3\bar{3}} + dz_{4\bar{4}}) + \frac{1}{2}(\bar{w}dz_{34} + w\,dz_{\bar{3}\bar{4}}))$. Therefore $X' = Y'_1 \times Y'_2$, where Y'_1 corresponds to (Y_1, ω'_1) and Y'_2 corresponds to (Y_2, ω'_2). So the rotated torus keeps being a product of two 2-tori.

8. Spin-rotation of a Weil abelian variety

Now we want to give explicitly a non-trivial example of a Spin-rotation of a complex 4-torus, starting with a Weil abelian variety X which is a product of two Weil abelian surfaces. The Spin-rotated complex torus turns out to be another Weil abelian variety X' which is not decomposable. Moreover, X' will be, from the arithmetic point of view, very different from the starting torus X.

We start by writing down explicitly the period matrix of a Weil abelian variety. Consider $K = \mathbb{Q}[\sqrt{-d}]$, $d > 0$ a square-free integer. Write (v_1, v_2, v_3, v_4) for the standard coordinates of $V = K^4 \otimes \mathbb{R}$. We consider the indefinite hermitian form $\tilde{h} = \frac{i}{2}(dv_{1\bar{1}} + dv_{2\bar{2}} - dv_{3\bar{3}} - dv_{4\bar{4}})$.

Take a basis for the subspaces $W_\pm$, which we write as the columns of the matrix

$$\begin{pmatrix} 1 & 0 & a' & e' \\ 0 & 1 & b' & f' \\ a & e & 1 & 0 \\ b & f & 0 & 1 \end{pmatrix}$$

(the first two columns are the basis of W_+ and the remaining two are the basis of W_-). We call this matrix the *defining matrix* of the Weil torus. For X to be an abelian four-fold, we have to take $W_+ \perp W_-$, w.r.t. $\tilde{h}$. In this case, the defining matrix is

$$\begin{pmatrix} 1 & 0 & \bar{a} & \bar{b} \\ 0 & 1 & \bar{e} & \bar{f} \\ a & e & 1 & 0 \\ b & f & 0 & 1 \end{pmatrix}. \tag{17}$$

Moreover, note that $I - AA^* > 0$, where $A = \begin{pmatrix} a & e \\ b & f \end{pmatrix}$. If we change to coordinates (w_1', w_2', w_3', w_4') in which the subspaces $W_\pm$ are the standard ones (by multiplying by the inverse of (17)), the lattice becomes generated (over $L = \mathbb{Z}[\sqrt{-d}]$) by the columns of

$$\begin{pmatrix} (I - A^*A)^{-1} & 0 \\ 0 & (I - AA^*)^{-1} \end{pmatrix} \begin{pmatrix} I & -A^* \\ -A & I \end{pmatrix}$$

The condition $W_+ \perp W_-$ gives that the metric is rewritten as $\tilde{h} = \frac{i}{2}(dw'_{1\bar{1}} + dw'_{2\bar{2}} - dw'_{3\bar{3}} - dw'_{4\bar{4}})$. By definition, $\tilde{h}$ is defined over K, so it takes values in $L = \mathbb{Z}[\sqrt{-d}]$ on Λ. Writing $\tilde{h} = \tilde{g} + i\omega$, we have that $\omega = \mathrm{Im}(\tilde{h})$ takes values in $\delta\,\mathbb{Z}$, $\delta = \sqrt{d}$, on the lattice Λ.

The lattice of a complex torus is defined up to multiplication by a matrix of $\mathrm{GL}(4,\mathbb{C})$ on the left. If we multiply by a matrix of the type $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$, we ensure that the subspaces $W_\pm$ are still the standard ones, but can arrange the lattice to be generated by the columns of

$$\begin{pmatrix} 1 & 0 & -\bar{a} & -\bar{b} \\ 0 & 1 & -\bar{e} & -\bar{f} \\ -a & -e & 1 & 0 \\ -b & -f & 0 & 1 \end{pmatrix}$$

and these elements multiplied by $\sqrt{-d} = i\delta$. Let (w_1, w_2, w_3, w_4) be these new coordinates. To go to the complex structure J, we need to conjugate the complex structure on W_-. So we take $z_3 = \bar{w}_3$, $z_4 = \bar{w}_4$. The lattice becomes

$$\begin{pmatrix} 1 & 0 & -\bar{a} & -\bar{b} \\ 0 & 1 & -\bar{e} & -\bar{f} \\ -\bar{a} & -\bar{e} & 1 & 0 \\ -\bar{b} & -\bar{f} & 0 & 1 \end{pmatrix}. \tag{18}$$

and the φ-transforms of those, i.e. the period matrix is

$$\Pi = \begin{pmatrix} 1 & i\delta & 0 & 0 & -\bar{a} & -i\delta\bar{a} & -\bar{b} & -i\delta\bar{b} \\ 0 & 0 & 1 & i\delta & -\bar{e} & -i\delta\bar{e} & -\bar{f} & -i\delta\bar{f} \\ -\bar{a} & i\delta\bar{a} & -\bar{e} & i\delta\bar{e} & 1 & -i\delta & 0 & 0 \\ -\bar{b} & i\delta\bar{b} & -\bar{f} & i\delta\bar{f} & 0 & 0 & 1 & -i\delta \end{pmatrix},$$

and $\varphi = \begin{pmatrix} i\delta & 0 & 0 & 0 \\ 0 & i\delta & 0 & 0 \\ 0 & 0 & -i\delta & 0 \\ 0 & 0 & 0 & -i\delta \end{pmatrix}$. This is the general form of the period matrix of a Weil abelian variety. We say that (18) is the *reduced period matrix* of X.

To continue, we shall put an extra condition to ensure that the Spin-rotated torus is an abelian variety. If ω is the Kähler form of X, then the Kähler form of the Spin-rotated torus is

$$\omega' = 2\frac{\omega+\gamma}{|\omega+\gamma|} = r\omega + s\,\gamma,$$

where $r, s \in \mathbb{R}$ satisfy $r^2+s^2=1$, taking $\gamma \in A_+$ with $|\gamma| = \sqrt{2}$.

In the coordinates (z_1, z_2, z_3, z_4) of X, we have the following choices obtained in (12) for performing a Spin-rotation,

$$\begin{aligned} \beta &= (dz_{12\bar{3}\bar{4}} + dz_{\bar{1}\bar{2}34}) + \omega^2, \\ \omega &= \frac{i}{2}(dz_{1\bar{1}} + dz_{2\bar{2}} + dz_{3\bar{3}} + dz_{4\bar{4}}), \\ \gamma &= \frac{1}{2}(dz_{12} + dz_{34} + dz_{\bar{1}\bar{2}} + dz_{\bar{3}\bar{4}}), \end{aligned} \tag{19}$$

In order for X' to be an abelian variety, we shall require that ω' be a rational class, which can be guaranteed if $\gamma = \mathrm{Re}(dz_{12}+dz_{34})$ is a rational class and $r, s \in \mathbb{Q}$, with $r^2+s^2=1$. For this, take the holomorphic symplectic form $c = dv_{12} + dv_{34}$ for $(V/\Lambda, I)$. This takes values in $\mathbb{Z}[\sqrt{-d}]$ on L. Now take $W_+ \perp W_-$ w.r.t. c. In terms of the matrix (17), this puts the extra condition $f = -\bar{a}, e = \bar{b}$, i.e.,

$$A = \begin{pmatrix} a & -\bar{b} \\ b & -\bar{a} \end{pmatrix}.$$

In this case A is an orthogonal matrix (i.e. it is in $\mathbb{R}_+ \cdot \mathrm{U}(2)$), so the metric $\tilde{h}$ only differs by a factor when changing from coordinates (w_1', w_2', w_3', w_4') to (w_1, w_2, w_3, w_4). Absorbing this factor $\nu = \det(I - AA^*) = 1 - |a|^2 - |b|^2$ in the metric, we have that $\tilde{h} = \frac{i}{2}(dw_{1\bar{1}} + dw_{2\bar{2}} - dw_{3\bar{3}} - dw_{4\bar{4}})$, and $\omega = \mathrm{Im}(\tilde{h})$ takes values in $\nu\delta\mathbb{Z}$.

For the class c, we have that $c = dw_{12} + dw_{34} = (1-|a|^2-|b|^2)(dv_{12}+dv_{34})$ takes values in νL. As $\gamma = \mathrm{Re}(c) = \mathrm{Re}(dw_{12}+dw_{34}) = \mathrm{Re}(dz_{12}+dz_{34})$, under $(z_1,z_2,z_3,z_4) = (w_1, w_2, \bar{w}_3, \bar{w}_4)$, we have that γ takes values in $\nu\mathbb{Z}$. So $\omega' = r\omega + s\gamma$ is rational (up to a multiple) when $r^2+s^2=1$, where $s = \delta\hat{s}$, $r, \hat{s} \in \mathbb{Q}$. This is equivalent to $r^2 + d\hat{s}^2 = 1$, $r, \hat{s} \in \mathbb{Q}$.

Summarizing, our starting Weil abelian torus has reduced period matrix (18)

$$\begin{pmatrix} 1 & 0 & -\bar{a} & -\bar{b} \\ 0 & 1 & -b & a \\ -\bar{a} & -b & 1 & 0 \\ -\bar{b} & a & 0 & 1 \end{pmatrix}.$$

In real coordinates, the period matrix is

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 & -a_1 & -a_2\delta & -b_1 & -b_2\delta \\ 0 & \delta & 0 & 0 & a_2 & -a_1\delta & b_2 & -b_1\delta \\ 0 & 0 & 1 & 0 & -b_1 & b_2\delta & a_1 & -a_2\delta \\ 0 & 0 & 0 & \delta & -b_2 & -b_1\delta & a_2 & a_1\delta \\ -a_1 & a_2\delta & -b_1 & -b_2\delta & 1 & 0 & 0 & 0 \\ a_2 & a_1\delta & -b_2 & b_1\delta & 0 & -\delta & 0 & 0 \\ -b_1 & b_2\delta & a_1 & a_2\delta & 0 & 0 & 1 & 0 \\ b_2 & b_1\delta & a_2 & -a_1\delta & 0 & 0 & 0 & -\delta \end{pmatrix}.$$

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$, $z_4 = x_4 + iy_4$ are the complex coordinates of X, and $z_1' = x_1' + iy_1'$, $z_2' = x_2' + iy_2'$, $z_3' = x_3' + iy_3'$, $z_4' = x_4' + iy_4'$ are the complex coordinates of the Spin-rotated torus X', the discussion in Section 7 implies that they are related by

$$\begin{pmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{pmatrix} = M \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}, \text{ and } \begin{pmatrix} x_3' \\ y_3' \\ x_4' \\ y_4' \end{pmatrix} = M \begin{pmatrix} x_3 \\ y_3 \\ x_4 \\ y_4 \end{pmatrix}, \text{ where } M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & s & 0 \\ 0 & -s & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 26. The change of variables relating (x_j, y_j) and (x_j', y_j') can be explicitly used to check the formula

$$\omega' = \frac{i}{2}(dz'_{1\bar{1}} + dz'_{2\bar{2}} + dz'_{3\bar{3}} + dz'_{4\bar{4}}),$$

and also to rewrite the class (19) in the new coordinates. The result is

$$\beta = (dz'_{12\bar{3}\bar{4}} + dz'_{\bar{1}\bar{2}34}) + (\omega')^2 .$$

Note that β is type $(2,2)$ w.r.t. J' as expected by Theorem 13.

The rotated Weil abelian variety X' has period matrix Π' obtained by multiplying Π by

$$\tilde{M} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}. \tag{20}$$

That is,

$$\Pi' = \left(\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & \delta r & s & 0 \\ 0 & -\delta s & r & 0 \\ 0 & 0 & 0 & \delta \\ -a_1 & a_2\delta & -b_1 & -b_2\delta \\ a_2 r - b_1 s & (a_1 r + b_2 s)\delta & -b_2 r + a_1 s & (b_1 r + a_2 s)\delta \\ -b_1 r - a_2 s & (b_2 r - a_1 s)\delta & a_1 r + b_2 s & (a_2 r - b_1 s)\delta \\ b_2 & b_1\delta & a_2 & -a_1\delta \end{matrix} \right.$$

$$\left.\begin{matrix} -a_1 & -a_2\delta & -b_1 & -b_2\delta \\ a_2r-b_1s & (-a_1r+b_2s)\delta & b_2r+a_1s & (-b_1r-a_2s)\delta \\ -a_2s-b_1r & (a_1s+b_2r)\delta & -b_2s+a_1r & (b_1s-a_2r)\delta \\ -b_2 & -b_1\delta & a_2 & a_1\delta \\ 1 & 0 & 0 & 0 \\ 0 & -\delta r & s & 0 \\ 0 & \delta s & r & 0 \\ 0 & 0 & 0 & -\delta \end{matrix}\right).$$

In complex terms, it is

$$\left(\begin{matrix} 1 & \delta ri & si & 0 \\ 0 & -\delta s & r & \delta i \\ -a_1+(a_2r-b_1s)i & a_2\delta+(a_1r+b_2s)\delta i & -b_1+(-b_2r+a_1s)i & -b_2\delta+(b_1r+a_2s)\delta i \\ -b_1r-a_2s+b_2i & (b_2r-a_1s)\delta+b_1\delta i & a_1r+b_2s+a_2i & (a_2r-b_1s)\delta-a_1\delta i \end{matrix}\right.$$

$$\left.\begin{matrix} -a_1+(a_2r-b_1s)i & -a_2\delta+(-a_1r+b_2s)\delta i & -b_1+(b_2r+a_1s)i & -b_2\delta+(-b_1r-a_2s)\delta i \\ -a_2s-b_1r-b_2i & (a_1s+b_2r)\delta-b_1\delta i & -b_2s+a_1r+a_2i & (b_1s-a_2r)\delta+a_1\delta i \\ 1 & -\delta ri & si & 0 \\ 0 & \delta s & r & -\delta i \end{matrix}\right).$$

To perform explicitly the Spin-rotation, we are going to consider the case where the reduced period matrix of X is

$$\begin{pmatrix} 1 & 0 & -\bar{a} & 0 \\ 0 & 1 & 0 & a \\ -\bar{a} & 0 & 1 & 0 \\ 0 & a & 0 & 1 \end{pmatrix}. \tag{21}$$

This means that $X = Y_1 \times Y_2$, where Y_1, Y_2 are two abelian surfaces of Weil type. The first abelian surface Y_1 corresponds to coordinates (z_1, z_3) and the second one Y_2, to coordinates (z_2, z_4).

Lemma 27. *The two surfaces Y_1, Y_2 are non-isomorphic, for generic a.*

Proof. Y_1 has defining matrix $\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ and Y_2 is given by $\begin{pmatrix} 1 & -\bar{a} \\ -\bar{a} & 1 \end{pmatrix}$. The endomorphism rings are $\mathrm{End}\,(Y) = \mathrm{End}\,(Y') = \mathbb{Q}[\sqrt{-d}]$.

Any possible isomorphism $\psi : Y \to Y'$ should interchange the endomorphisms φ, up to sign. In particular, ψ either preserves or permutes the two eigenspaces of φ. Suppose the first case (the second one is analogous). Let B be the matrix of ψ w.r.t. the coordinates of K^2. Then

$$\begin{pmatrix} 1 & -\bar{a} \\ -\bar{a} & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} = B \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}.$$

where B is a matrix with coefficients in $K=\mathbb{Q}[\sqrt{-d}]$, $\epsilon_1,\epsilon_2\in\mathbb{C}$. Hence we have an equation

$$-\bar{a}=\frac{\gamma+\delta a}{\alpha+\beta a},$$

for $\alpha,\beta,\gamma,\delta\in K$. This cannot happen for general a. $\square$

For the Weil abelian surface Y_1, we have a Weil class, which is of the form $\alpha_1 = 2\,\mathrm{Re}(dw_{13}) = 2\,\mathrm{Re}(dz_{1\bar{3}}) = dz_{1\bar{3}} + dz_{\bar{1}3}$. In this situation, the Weil class is a rational $(1,1)$-class. By the Lefschetz theorem, there is a (rational) divisor D_1 and a corresponding holomorphic line bundle $\mathcal{L}_1=\mathcal{O}(D_1)$ over Y with $c_1(\mathcal{L}_1)=[D_1]=\alpha_1$ (after multiplying by a large integer if necessary). Note that α_1 is primitive, so $\alpha_1\wedge\omega_1=0$.

For Y_2 we have a second Weil class $\alpha_2=-dz_{2\bar{4}}-dz_{\bar{2}4}$ and a divisor D_2 and line bundle $\mathcal{L}_2=\mathcal{O}(D_2)$ with $c_1(\mathcal{L}_2)=[D_2]=\alpha_2$. The rank 2 bundle

$$E=\mathcal{L}_1\oplus\mathcal{L}_2$$

is polystable, since the degrees of both summands are zero (w.r.t. the Kähler form $\omega=\omega_1+\omega_2$). We compute

$$4\beta(E)=-(D_1-D_2)^2=-2dz_{1\bar{1}3\bar{3}}-2dz_{2\bar{2}4\bar{4}}+2dz_{12\bar{3}\bar{4}}+2dz_{\bar{1}\bar{2}34}+2dz_{1\bar{2}\bar{3}4}+2dz_{\bar{1}23\bar{4}}.$$

Starting with $\alpha_1'=-2\mathrm{Im}(dz_{1\bar{3}})=i\,dz_{1\bar{3}}-i\,dz_{\bar{1}3}$ and $\alpha_2'=i\,dz_{2\bar{4}}-i\,dz_{\bar{2}4}$, we get a bundle $E'=\mathcal{L}_1'\oplus\mathcal{L}_2'$ such that

$$4\beta(E')=-2dz_{1\bar{1}3\bar{3}}-2dz_{2\bar{2}4\bar{4}}+2dz_{12\bar{3}\bar{4}}+2dz_{\bar{1}\bar{2}34}-2dz_{1\bar{2}\bar{3}4}-2dz_{\bar{1}23\bar{4}}.$$

The direct sum $E\oplus E'$ has

$$\beta(E\oplus E')=-dz_{1\bar{1}3\bar{3}}-dz_{2\bar{2}4\bar{4}}+dz_{12\bar{3}\bar{4}}+dz_{\bar{1}\bar{2}34}.$$

Note that this class has $k=\frac{2}{3}$ and it is Spin-rotable via

$$c\in\langle dz_{12}+dz_{34},dz_{13}+dz_{24},i\,dz_{13}-i\,dz_{24}\rangle.$$

Let $F=\mathrm{End}\,(E\oplus E')$ be the SU-bundle associated to $E\oplus E'$.

Now take $E''=\mathcal{O}(H_1)\oplus\mathcal{O}(H_2)$, where H_1,H_2 are the divisors corresponding to ω_1,ω_2, respectively. It is polystable w.r.t. $\omega=\omega_1+\omega_2$ (corresponding to the polarisation $H=H_1+H_2$), and it has

$$4\beta(E'')=-(H_1-H_2)^2=2dz_{1\bar{1}3\bar{3}}+2dz_{2\bar{2}4\bar{4}}-2dz_{1\bar{1}2\bar{2}}-2dz_{1\bar{1}4\bar{4}}-2dz_{2\bar{2}3\bar{3}}-2dz_{3\bar{3}4\bar{4}}.$$

This has $k=\frac{1}{3}$, and it is Spin-rotable via

$$c\in\langle dz_{12}+dz_{34},i\,dz_{12}-i\,dz_{34},dz_{14}+dz_{23},i\,dz_{14}-i\,dz_{23}\rangle.$$

Let $F'=\mathrm{End}\;E''$.

Consider the sum $F\oplus F'$. This is polystable and it has

$$\beta=-\frac{1}{2}\sum_{i<j}dz_{i\bar{i}j\bar{j}}+dz_{12\bar{3}\bar{4}}+dz_{\bar{1}\bar{2}34}=\beta_0+\omega^2\,.$$

Moreover, it is Spin-rotable with $c = dz_{12} + dz_{34}$.

Now starting with X with reduced period matrix (21), the rotated lattice is

(22)
$$\left(\begin{array}{cccccccc} 1 & \delta ri & si & 0 & -a_1 + a_2 ri & -a_2\delta - a_1 r\delta i & a_1 si & -a_2 s\delta i \\ 0 & -\delta s & r & \delta i & -a_2 s & a_1 s\delta & a_1 r + a_2 i & -a_2 r\delta + a_1 \\ -a_1 + a_2 ri & a_2\delta + a_1 r\delta i & a_1 si & a_2 s\delta i & 1 & -\delta ri & si & 0 \\ -a_2 s & -a_1 s\delta & a_1 r + a_2 i & a_2 r\delta - a_1\delta i & 0 & \delta s & r & -\delta i \end{array}\right.$$

The endomorphism

$$\varphi = \delta \begin{pmatrix} ri & s & 0 & 0 \\ -s & -ri & 0 & 0 \\ 0 & 0 & -ri & -s \\ 0 & 0 & s & ri \end{pmatrix} \tag{23}$$

leaves the lattice fixed, and $\varphi^2 = -d\,\mathrm{Id}$, so it gives the structure of an L-module. A basis of the lattice, as an L-module, is given by the first, third, fifth and seventh vectors, i.e.

$$\widehat{\Pi} = \begin{pmatrix} 1 & si & -a_1 + a_2 ri & a_1 si \\ 0 & r & -a_2 s & a_1 r + a_2 i \\ -a_1 + a_2 ri & a_1 si & 1 & si \\ -a_2 s & a_1 r + a_2 i & 0 & r \end{pmatrix}.$$

The diagonalizing vectors of φ are given by the columns of the matrix

$$P = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{r+1}{2}} & 0 & i\sqrt{\frac{r-1}{2}} & 0 \\ \frac{si}{\sqrt{2(r+1)}} & 0 & -\frac{s}{\sqrt{2(r-1)}} & 0 \\ 0 & \sqrt{\frac{r-1}{2}} & 0 & i\sqrt{\frac{r+1}{2}} \\ 0 & \frac{si}{\sqrt{2(r-1)}} & 0 & -\frac{s}{\sqrt{2(r+1)}} \end{pmatrix}, \tag{24}$$

where we have taken the vectors to be an orthonormal basis, so that the metric has still the same (standard) form.

Now we parametrize $r^2 + s^2 = 1$ with a single variable $x \in \mathbb{R}$, by taking

$$r = \frac{1 - x^2}{1 + x^2}, \quad s = \frac{2x}{1 + x^2}. \tag{25}$$

Recall that $s = \delta\hat{s}$, where $\delta = \sqrt{d}$, so that the change of variables $x = y/\delta$ yields

$$r = \frac{d - y^2}{d + y^2}, \quad \hat{s} = \frac{2y}{d + y^2},$$

for $r^2 + d\hat{s}^2 = 1$, and $r, \hat{s} \in \mathbb{Q} \iff y \in \mathbb{Q}$.

In the basis where φ has a standard form, the lattice is generated by the columns of

$$P^{-1}\widehat{\Pi} = \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} 1 & xi & -\bar{a} & x\bar{a}i \\ aix & a & -xi & 1 \\ -x & -i & xa & -ia \\ \bar{a}i & \bar{a}x & -i & x \end{pmatrix}.$$

Therefore $I = \frac{1}{\sqrt{d}}\varphi$ is the standard complex structure, and the subspaces $W_\pm$ are the standard ones, $W_+ = \langle z_1, z_2\rangle$, $W_- = \langle z_3, z_4\rangle$, with respect to the lattice obtained by conjugating the last two rows:

$$C = \begin{pmatrix} 1 & xi & -\bar{a} & x\bar{a}i \\ aix & a & -xi & 1 \\ -x & i & x\bar{a} & i\bar{a} \\ -ai & ax & i & x \end{pmatrix} \tag{26}$$

(we are ignoring the overall scalar factor).

We want to put this lattice into standard form, to see where the subspaces $W_\pm$ go. For this, we have to multiply by a matrix in $U(2,2)$. If we modify the lattice, by taking a different basis, this gets easier. Consider

$$f = \frac{2ix}{1+x^2} = \frac{2i\delta y}{d+y^2}, \qquad g = \frac{1-x^2}{1+x^2} = \frac{d-y^2}{d+y^2},$$

both of which are in $K = \mathbb{Q}[\sqrt{-d}]$, when $y \in \mathbb{Q}$. Consider the new basis for the lattice

$$\hat{C} = C \begin{pmatrix} 1 & 0 & -f & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & g \\ 0 & 1 & 0 & f \end{pmatrix} = \begin{pmatrix} 1 & x\bar{a}i & -ix & -2\bar{a} \\ aix & 1 & a & ix \\ -x & i\bar{a} & -i & -x\bar{a} \\ -ai & x & -ax & i \end{pmatrix}.$$

As we change the basis by using rational numbers, the resulting torus is actually a torus isogenous to the given one.

Now let us restrict to $x \in (0,1)$. We see that $\hat{C} \in U(2,2)$. To make the lattice standard, we just conjugate by (forgetting again an overall factor)

$$\hat{C}^{-1} = \begin{pmatrix} 1 & -i\bar{a}x & -x & -i\bar{a} \\ -iax & 1 & -ia & -x \\ a & ix & ax & -i \\ -ix & -\bar{a} & i & -x\bar{a} \end{pmatrix}.$$

Note that the column vectors are orthogonal w.r.t. the indefinite metric $\tilde{h}$.

This is conjugated, via an $U(2,2)$-transformation on the right of the form $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$, to

$$\begin{pmatrix} I & \bar{B}^T \\ B & I \end{pmatrix},$$

where

$$B = \frac{1}{1+x^2|a|^2} \begin{pmatrix} a(1-x^2) & ix(1+|a|^2) \\ -ix(1+|a|^2) & -\bar{a}(1-x^2) \end{pmatrix}.$$

This corresponds to an abelian variety of Weil type for

$$\tilde{a} = \frac{a(1-x^2)}{1+x^2|a|^2}, \quad \tilde{b} = -i\frac{x(1+|a|^2)}{1+x^2|a|^2}.$$

Theorem 13 says that the class β is an algebraic cycle for the Weyl abelian variety X'.

Which type of abelian variety is X'? Recall that we are taking $y \in \mathbb{Q}$. This guarantees that X' is an abelian variety. The endomorphism ring of X is $\mathrm{End}\,(X) = \mathrm{End}\, Y \times \mathrm{End}\, Y' = K \times K$, by Lemma 27. Now we want to compute $\mathrm{End}\,(X')$.

Doing the change of variables $x = y/\delta$, we get

$$\begin{aligned} \tilde{a} &= \frac{a(d-y^2)}{d+y^2|a|^2}, \\ \tilde{b} &= -i\delta\,\frac{y(1+|a|^2)}{d+y^2|a|^2} = i\varpi\,. \end{aligned} \tag{27}$$

A pair $(\tilde{a},\varpi) \in \mathbb{C}\times\mathbb{R}$ can appear as in (27) if they satisfy the equation

$$q(1+|\tilde{a}|^2+\varpi^2)+i\varpi = 0, \tag{28}$$

where $q = \frac{y\delta i}{y^2+d}$. Note that $y \in \mathbb{Q}$ implies that $q \in K$.

Equation (28) gives a smoothly varying family of Kähler 4-tori (of Weil type) for all real values $y \in (0,\delta)$, although we only know that β is HYM for $y \in \mathbb{Q}$. Let X' be the Kähler 4-torus for given (a,ϖ,y) satisfying (28). The groups $\mathrm{End}\,(X')$ form a family. In particular, for small y we have that $\dim \mathrm{End}\,(X') \leq 4$.

Let ψ be an endomorphism of X'. Suppose that ψ commutes with the action of φ. Then either it preserves or it swaps the $\pm i\delta$-eigenspaces. Suppose the first case. The argument of Lemma 27 says that there is some matrix $M \in \mathrm{GL}(4,K)$ such that the defining matrix satisfies

$$\begin{pmatrix} I & \bar{B}^T \\ B & I \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & P_4 \end{pmatrix} \begin{pmatrix} I & \bar{B}^T \\ B & I \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}.$$

This gets rewritten

$$(M_3 + M_4 B)(M_1 + M_2 B)^{-1} = B$$

or equivalently,

$$BM_2B + BM_1 - M_4B - M_3 = 0. \tag{29}$$

In our situation, we have the equation (28). For fixed q, we take $(\tilde{a}, \varpi)$ as generic as possible within this constraint. More explicitly, writing $\tilde{a} = a_1 + ia_2$, we ask that any two of a_1, a_2, ϖ are algebraically independent, and also the three a_1, a_2, ϖ are K-independent. An easy calculation shows that under these circunstances, (29) implies that

$$M = \left(\begin{array}{cc|cc} \alpha & 0 & 0 & \gamma \\ 0 & \beta & \gamma & 0 \\ \hline 0 & \gamma & \alpha & 0 \\ \gamma & 0 & 0 & \beta \end{array} \right), \tag{30}$$

where $\gamma = q\alpha - q\beta$. This produces a vector space over K of dimension 2. Hence this constitutes the full endomorphism ring, and it is so for any value of y (even not generic ones).

The endomorphism ring of X' is generated by two elements, Id and

$$F = \begin{pmatrix} 1 & 0 & 0 & 2q \\ 0 & -1 & 2q & 0 \\ 0 & 2q & 1 & 0 \\ 2q & 0 & 0 & -1 \end{pmatrix}.$$

We have that $F^2 = (4q^2 + 1)\,\mathrm{Id}$. Therefore $\mathrm{End}\,(X') = \mathbb{Q}[\sqrt{-d}, \sqrt{4q^2+1}]$, which is an imaginary quadratic extension of a degree 2 totally real field. So X' is of type $VI(4,1)$ in the classification of abelian varieties (see Appendix B in [7]). For these abelian varieties, all Hodge classes are algebraic, because they are products of divisors.

Discussion. In Section 8 we have given an example of a Weil abelian variety X for which there is a (poly)stable vector bundle $E \to X$ which is Spin-rotable. This produced another Weil abelian variety X' with another (poly)stable vector bundle whose second Chern class has a non-zero component Weil class. Therefore, such Weil class is algebraic. The fact that it is a product of divisors has a geometric explanation: the Spin(7)-connection on the bundle is actually decomposable (E splits as a direct sum of line bundles with Spin(7)-connections), therefore the same must happen for the (poly)stable bundle on X'.

Nonetheless, the Spin-rotations relate abelian varieties of very different geometric and arithmetic nature. This can be used to produce stable bundles on abelian varieties and hence to prove algebraicity of Hodge classes. For obtaining deeper consequences, one should start with non-decomposable stable bundles. Moreover, these

Spin-rotations produce also Kähler non-algebraic tori (e.g. when $y \notin \mathbb{Q}$ in the situation of Section 8). Therefore the construction is useful to produce stable bundles on Kähler tori.

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